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Semiinvariants, Senatov Moments and Density Decomposition

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Abstract
It is proposed to introduce into Probability Theory courses such a new moment characteristic of random variable as Senatov moment. Naturalness of this proposal is confirmed by three views of appearance of Senatov moments. Introducing of them will answer the question about what is analogue of Taylor series of function for density.

Keywords: moments, semiinvariants, Senatov moments, Fourier transform, Fourier series, density decomposition

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Краткое сообщение

Семиинварианты, моменты Сенатова и разложение плотности

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Аннотация
Предлагается ввести в программы курсов теории вероятностей рассмотрение относительно новой моментной характеристики случайных величин — моментов Сенатова. Естественность этого предложения подтверждается тремя взглядами на возникновение моментов Сенатова, а их введение позволит ответить на вопрос, что является аналогом ряда Тейлора функции для плотности.

Ключевые слова: моменты, семиинварианты, моменты Сенатова, преобразование Фурье, ряд Фурье, разложение плотности

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Introduction. In probability theory courses, in addition to the usual moments of a random variable ξ:

$$\alpha_k = M\xi^k = \int_{-\infty}^{\infty} x^k dF(x), k \in Z_+,$$

where $F(x)$ is the distribution function of the random variable under consideration, other moment characteristics are also described, for example,
absolute moments:

$$M\,|\xi|^k,$$

central moments:

$$M\,(\xi - M\,\xi)^k$$

and factorial

$$M\xi^{[k]} = M\,\xi(\xi-1)\ldots(\xi-k+1)$$

moments.

The central problem of probability theory — the central limit theorem — has led in its development to the appearance of two more moment characteristics, called semiinvariants, which are told to mathematics students, and Senatov moments, which are just beginning to enter the course programs.

The apparatus for determining moments proposed by V. V. Senatov can also be used in applied problems, for example, when calculating the coefficients of turbulent exchange for the equations of hydrodynamics of systems with a free surface, including marine and coastal [1].

Aim of work. Senatov's moments deserve to be included in the programs of probability theory courses. In the paper, this will be justified with the help of three questions, at first glance, unrelated to each other.

For the sake of simplicity, we will consider the random variables considered in this paper to be centered, normalized and absolutely continuous, which have moments of all natural orders, and the characteristic function:

$$f(t) = Me^{it\xi}$$

is represented by Taylor series and is absolutely integrable.

The first question. The even moments of a standard normal random variable increase and increase rapidly — the standard normal moment of the order of $2k$ is $(2k-1)!!$, $k \in N$ (in the future we will consider k a non-negative integer, unless otherwise specified). But investigating the convergence of centered and normalized convolutions to a standard normal random variable by investigating the convergence of a numerical sequence to a non-zero number is usually technically more difficult than investigating convergence to zero. Accordingly, it becomes necessary to introduce new natural moment characteristics, which for a standard normal random variable will be zero, with the exception, perhaps, of the most initial orders. Since the moments are related to the derivatives of the characteristic function by equality:

$$i^k\alpha_k = (f(t))^{(k)}{}_{t=0},$$

and the standard normal characteristic function is exp(-t2/2), then it is necessary to propose such a transformation of the latter so that the derivatives of the resulting composition at zero *quickly* become zero.

The use of logarithm as the first such transformation was proposed in 1889 by the Danish astronomer and mathematician Thorvald Nicolai Thiele, calling the obtained characteristics semiinvariants:

$$i^k\kappa_k = (\ln(f(t)))^{(k)}{}_{t=0}\,.$$

Indeed, $\ln(\exp(-t^2/2)) = -t^2/2$, тогда $\kappa_0 = 0$, $\kappa_1 = 0$, $\kappa_2 = 1$ и $\kappa_k = 0$ при $k > 3$.

An important exceptional property of semiinvariants: since the characteristic function of the convolution is equal to the product of the characteristic functions of the summands, the semiinvariants of the convolution are equal to the sum of the semiinvariants of the summands. But working with a complex logarithm requires special care and is often associated with significant technical difficulties.

Another transformation is no less natural — in 2001, Vladimir Vasilyevich Senatov, Professor of the Department of Probability Theory of the Faculty of Mechanics and Mathematics of the Lomonosov Moscow State University, finally determined the characteristics called Senatov moments from 2021 (after his death):

$$i^k\theta_k = (\exp(t^2/2)f(t))^{(k)}{}_{t=0}\,.$$

All the Senatov moments of a standard normal random variable are zero, except $\theta_0 = 1$.

All the mentioned moment characteristics of an arbitrary random variable exist or do not exist simultaneously, always $\kappa_0 = 0$, $\theta_0 = 1$, for centered and normalized random variables $\kappa_1 = \theta_1 = \theta_2 = 0$, $\kappa_2 = 1$.

The second question is related to the fact that the representation of a characteristic function by its Taylor series:

$$f(t) = \sum_{k=0}^{\infty} \frac{\alpha_k}{k!}(it)^k$$

does not allow us to find the density $p(x)$ through the inversion formula:

$$p(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}f(t)dt,$$

since the Fourier transform of the power function does not exist.

It is possible to get out of the impasse with the help of semiinvariants and Senatov moments. To do this, recall that the Chebyshev-Hermite polynomials, which are eigenfunctions of the Schrodinger equation [2–3] and form an orthogonal system on the set of real numbers with the weight of the standard normal density $\varphi(x)=e^{-x^2/2}/\sqrt{2\pi}$:

$$H_k(x)=(-1)^k e^{x^2/2}\left(e^{-x^2/2}\right)^{(k)}$$

have the property (1):

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{-itx}(it)^k e^{-t^2/2}dt=H_k(x)\varphi(x).$$

For a characteristic function , the representation is valid:

$$f(t)=\exp(\mathrm{In}(f(t)))=\exp\left(\sum_{k=0}^{\infty}\frac{\kappa_k}{k!}(it)^k\right)=e^{-t^2/2}\exp\left(\sum_{k=3}^{\infty}\frac{\kappa_k}{k!}(it)^k\right),$$

and therefore property (1) allows, by presenting its second exponent with a Taylor series, to apply the inversion formula.

Similar arguments using the Senatov moments allow us to answer the third question — which analogue of the Taylor series of the function can be proposed for a random variable in the face of its density. Since:

$$f(t)=e^{-t^2/2}\left(e^{t^2/2}f(t)\right)=e^{-t^2/2}\left(\sum_{k=o}^{\infty}\frac{\theta_k}{k!}(it)^k\right)=e^{-t^2/2}\left(1+\sum_{k=3}^{\infty}\frac{\theta_k}{k!}(it)^k\right),$$

that property (1) allows you to immediately apply the inversion formula and obtain the expansion for the density in the form of the corresponding Fourier series:

$$p(x)=\varphi(x)+\sum_{k=3}^{\infty}\frac{\theta_k}{k!}H_k(x)\varphi(x).$$

For centered and normalized sums of n independent random variables, such expansions are called asymptotic [4], since all summands under the sign of the sum will tend to zero with the growth of n, which follows from the expression of the moments of convolutions through the moments of the initial distribution:

$$\frac{\theta_k(F_n)}{k!}=\sum\frac{n!}{j_0!\,j_3!\ldots j_n!}\left(\frac{\theta_3}{3!n^{\frac{3}{2}}}\right)^{j_3}\cdots\left(\frac{\theta_k}{k!n^{\frac{k}{2}}}\right)^{j_k},$$

where summation is performed over whole non-negative sets:

$$j_0+j_3+j_4+\ldots j_k=n,\ 3j_3+4j_4+\ldots kj_k=k.$$

The speed of striving for zero “triples” is interesting (Table 1):

$$\theta_k(F_n)=O\left(n-\frac{\left[\frac{k}{3}+3\left\{\frac{k}{3}\right\}\right]}{2}\right),n\to\infty\ .$$

Table 1

The rate of striving for zero of the “triples” of terms

k	3	<u>4</u>	5	6	<u>7</u>	8	9	<u>10</u>	11	…
$\frac{\left[\frac{k}{3}\right]+3\left\{\frac{k}{3}\right\}}{2}$	0.5	<u>1</u>	1.5	1	<u>1.5</u>	2	1.5	<u>2</u>	2.5	…

For example:

$$\theta_3(F_n)=\frac{\theta_3}{\sqrt{n}},\ \theta_4(F_n)=\frac{\theta_4}{n},\ \theta_5(F_n)=\frac{\theta_5}{n^{1,5}},\ \theta_6(F_n)=\frac{6!}{n^2}\left(\frac{n-1}{2!}\frac{\theta_3^2}{3!^2}+\frac{\theta_6}{6!}\right).$$

The *magical* connection of the Senatov moments with Chebyshev-Hermite polynomials is the following equality:

$$\theta_k=\int_{-\infty}^{\infty}H_k(x)\,dF(x).$$

V. V. Senatov defined them this way in 2001 [5], calling them Chebyshev-Hermite moments because of this connection, therefore, in literature and research until last year they are found and used under this name. Now, as a sign of memory of an outstanding scientist who worked at Moscow University and proposed, in particular, asymptotic expansions with an explicit accuracy estimate that can be brought to numerical values, we will call them Senatov moments [6].

Conclusion. The use of Senatov moments made it possible to advance the task of studying the convergence rate in the central limit theorem so qualitatively that these moment characteristics began to be perceived very naturally. This allows, according to the authors, to raise the question of their inclusion in the program of those courses in probability theory, where the central limit theorem is proved by the method of characteristic functions. And for students of mathematics, it is simply necessary to do this in order to prepare for the study of a special course “Additional chapters of probability theory”.

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