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Original article

Symmetrized Versions of the Seidel and Successive OverRelaxation Methods for Solving Two-Dimensional Difference Problems of Elliptic Type

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Abstract

Introduction. This article is devoted to the consideration of options for symmetrization of two-layer implicit iterative methods for solving grid equations that arise when approximating boundary value problems for two-dimensional elliptic equations. These equations are included in the formulation of many problems of hydrodynamics, hydrobiology of aquatic systems, etc. Grid equations for these problems are characterized by a large number of unknowns — from 10^6 to 10^{10}, which leads to poor conditionality of the corresponding system of algebraic equations and, as a consequence, to a significant increase in the number of iterations, necessary to achieve the specified accuracy. The article discusses a method for reducing the number of iterations for relatively simple methods for solving grid equations, based on the procedure of symmetrized traversal of the grid region.

Materials and Methods. The methods for solving grid equations discussed in the article are based on the procedure of symmetrized traversal along the rows (or columns) of the grid area.

Results. Numerical experiments have been performed for a model problem — the Dirichlet difference problem for the Poisson equation, which demonstrate a reduction in the number of iterations compared to the basic algorithms of these methods.

Discussion and Conclusions. This work has practical significance. The developed software allows it to be used to solve specific physical problems, including as an element of a software package.

Keywords: two-dimensional problem of elliptic type, iterative methods, relaxation methods, complete relaxation method, Seidel method, upper relaxation method

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Научная статья

Симметризованные варианты методов Зейделя и верхней релаксации решения двумерных разностных задач эллиптического типа

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Аннотация

Введение. Данная статья посвящена рассмотрению вариантов симметризации двухслойных неявных итерационных методов для решения сеточных уравнений, возникающих при аппроксимации краевых задач для двумерных уравнений эллиптического типа. Данные уравнения входят в постановки многих задач гидродинамики, гидробиологии водных систем и др. Сеточные уравнения для данных задач характеризуются большим количеством неизвестных — от 10^6 до 10^{10}, что приводит к плохой обусловленности соответствующей системы алгебраических уравнений и, как следствие, к существенному росту числа итераций, необходимых для достижения заданной точности. В статье рассмотрен метод снижения числа итераций для относительно простых методов решения сеточных уравнений (метода Зейделя и верхней релаксации).

Материалы и методы. Рассматриваемые в статье методы решения сеточных уравнений базируются на процедуре симметризованного обхода по строками (или столбцами) сеточной области.

Результаты исследования. Выполнены численные эксперименты для модельной задачи — разностной задачи Дирихле для уравнения Пуассона, которые демонстрируют сокращение числа итераций по сравнению с базовыми алгоритмами данных методов.

Обсуждение и заключения. Данная работа имеет практическую значимость. Разработанное программное средство позволяет его использовать для решения конкретных физических задач, в том числе как элемента программного комплекса.

Ключевые слова: двумерная задача эллиптического типа, итерационные методы, релаксационные методы, метод полной релаксации, метод Зейделя, метод верхней релаксации

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Introduction. In numerical modeling of technical systems, physical phenomena and technological processes, as a rule, a significant proportion of the total amount of computational work is the solution of systems of linear algebraic equations (SLAE) that arise when the corresponding differential or integro-differential equations are discretized. A special class is represented by systems of linear algebraic equations with symmetric positive definite matrices. Depending on the proposed approach to constructing the next iterative approximation, several iterative methods for solving these SLAE are distinguished [1–3]. Among them are the methods of Seidel and Successive OverRelaxation. The popularity of these methods can be explained by their simplicity and wide popularity among researchers [4]. In this regard, there is a natural interest in studying various variants of the methods under consideration and the desire to obtain the advantages of using them.

In this article, variants of symmetrization of Seidel and Successive OverRelaxation methods for solving two-dimensional difference problems of elliptic type are considered. Based on the results of numerical calculations of the solution of the Dirichlet problem for the Poisson equation in a rectangular area, a reduction in the number of iterations compared to the basic algorithms of these methods is demonstrated. The table, which shows the dependences of the number of iterations on the number of grid nodes of the computational domain when using the methods under consideration, makes it possible to visually verify that the symmetrized version of the Successive OverRelaxation method can significantly reduce the required number of iterations to achieve a given accuracy and, as a result, reduce the calculation time.

Materials and methods

1. Seidel and Successive OverRelaxation methods. In a finite-dimensional Hilbert space , the problem of finding a solution to an operator equation is considered:

$$Ax = f, \quad A: \ H \to H, \tag{1}$$

where A is the linear operator, x is the desired function, f is the known function of the right part.

To find a solution to problem (1), we will use an implicit two-layer iterative scheme:

$$B\frac{y^{k+1} - y^k}{\tau_{k+1}} + Ay^k = f, \ B: \ H \to H, \ k = 0, 1, 2, ..., \tag{2}$$

with an arbitrary approximation $y^0 \in H$.

Equation (2) uses the notation: B is some invertible operator; k is iteration number; y^k is the vector of the k-th iterative approximation; τ_{k+1} is the iterative parameter, $\tau_{k+1} > 0$.

To represent the Seidel iterative method in matrix form, we write the matrix as a sum of diagonal, lower triangular and upper triangular matrices:

$$A = D + L + U, \tag{3}$$

where:

$$D = \begin{pmatrix} a_{11} & 0 & ... & 0 & 0 \\ 0 & a_{22} & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & a_{N-1N-1} & 0 \\ 0 & 0 & ... & 0 & a_{NN} \end{pmatrix}, \ L = \begin{pmatrix} 0 & 0 & ... & 0 & 0 \\ a_{21} & 0 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ a_{N-11} & a_{N-12} & ... & 0 & 0 \\ a_{N1} & a_{N2} & ... & a_{NN-1} & 0 \end{pmatrix},$$

$$U = \begin{pmatrix} 0 & a_{12} & ... & a_{1N-1} & a_{1N} \\ 0 & 0 & ... & a_{2N-1} & a_{2N} \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & 0 & a_{N-1N} \\ 0 & 0 & ... & 0 & 0 \end{pmatrix}.$$

Denote by $y^k = \left(y_1^{(k)} y_2^{(k)}, ..., y_N^{(k)}\right)$ the vector of the k-th iterative approximation.

Using expression (3), we write the Seidel method in the form:

$$(D + L)y^{k+1} + Uy^k = f, \ k = 0, 1, 2, ... \tag{4}$$

Bringing the iterative scheme (4) to the canonical form of two-layer iterative schemes (2), we find:

$$(D + L)\left(y^{k+1} - y^k\right) + Ay^k = f, \ k = 0,1,2,... , \ y^0 \in H. \tag{5}$$

When comparing schemes (2) and (5), it can be seen that they will be identical at $B = D + L$, $\tau_{k+1} = 1$. Scheme (5), as well as scheme (2), will be implicit, and the operator is not self-adjoint in space H (here the operator B corresponds to the lower triangular matrix).

To accelerate the convergence of the Seidel method, it is modified by introducing a numerical parameter ω, into the iterative scheme (5), so that:

$$(D + \omega L)\frac{y^{k+1} - y^k}{\omega} + Ay^k = f, \ k = 0,1,2,... , \ y^0 \in H. \tag{6}$$

For scheme (6), the iterative method is the Successive OverRelaxation, (SOR).

The identity of schemes (6) and (2) can be observed at $B = D + \omega L$, $\tau_{k+1} = \omega$. As in the case of using the Seidel method, the matrix corresponds to the lower triangular matrix. Therefore, the introduction of the parameter ω does not take us out of the class of triangular iterative methods. The implementation of one iterative step of the scheme (6) is carried out with approximately the same cost of arithmetic operations as in the scheme (5).

Sufficient conditions for convergence of the considered schemes (5), (6) are self-conjugacy and positive definiteness of the operator A in space H [5]. In the following statement, we assume these conditions for the operator A to be fulfilled.

2. Symmetrization of Seidel and Successive OverRelaxation methods. Consider the Dirichlet difference problem for an elliptic equation. For simplicity, let's take the Dirichlet problem for the Poisson equation.

Let on a rectangular grid:

$$\omega_h = \left\{ x_{ij} = (ih_1, jh_2),\ i = 1,...,N_1,\ j = 1,...,N_2,\ h_\alpha = \frac{l_\alpha}{N_\alpha},\ \alpha = 1,2 \right\},$$

entered in a rectangle $\overline{G} = \{0 \le x_\alpha \le l_\alpha,\ \alpha = 1,2\}$, is required to find a solution to the difference problem:

$$\begin{cases} \sum_{\alpha=1}^{2} y_{\bar{x}_\alpha x_\alpha} = f(x), & x \in \omega_h, \\ y(x) = g(x), & x \in \gamma, \end{cases} \tag{7}$$

where $f(x)$ and $g(x)$ are the given functions, γ is the boundary of the grid ω_h, $\omega_h = \overline{\omega}_h \setminus \gamma$.

When solving system (7) by the Seidel method or the Successive OverRelaxation method, calculations begin at a point $x_{N_1N_2} = x_{ij}(i = N_1, j = N_2)$ and are carried out along the rows or columns of the calculated grid ω_h to a point $x_{N_1N_2} = x_{ij}(i = N_1, j = N_2)$ (the image of the grid layout is not given due to the evidence of its representation). Then the calculations start from the starting point and repeat until a solution is reached. The main idea of symmetrization of iterative methods is to add a new solution vector. Here, after the calculation is made from point x_{11} to point x_{N1N2}, it will continue further from point x_{N1N2} to point x_{11} in reverse order along the columns or rows of the grid and then repeat from the starting point.

We construct symmetrized variants of the Seidel and Successive OverRelaxation methods under the assumption that a rectangular grid with equal numbers of nodes in each of the coordinate directions is used. Let C be a matrix of permutations of size $N \times N$ ($N = N_1 = N_2$), of the form:

$$C = \begin{pmatrix} 0 & 0 & ... & 0 & a_{1N} \\ 0 & 0 & ... & a_{2N-1} & 0 \\ ... & ... & ... & ... & ... \\ 0 & a_{N-12} & ... & 0 & 0 \\ a_{N1} & 0 & ... & 0 & 0 \end{pmatrix}.$$

The iterative scheme (5) as a result of the symmetrization of the Seidel method takes the form:

$$(D + L)(y^{k+1} - y^k) + Ay^k = f,\ y^0 \in H,\ k = 0,1,2,...,k_1, \tag{8}$$

$$(D + L)C(y^{k+1} - y^k)C + ACy^kC = f,\quad y^0 = y^{k_1},\ k = k_1 + 1, k_1 + 2,... . \tag{9}$$

Sufficient convergence conditions of schemes (8)–(9) for the symmetrized Seidel method are determined from the constraints imposed on the operator (as mentioned earlier, this is a self-adjoint and positive definite operator).

Let's proceed to the construction of Symmetric Successive OverRelaxation (SSOR).

The iterative scheme (6) as a result of symmetrization takes the form:

$$(D + \omega L)\frac{y^{k+1} - y^k}{\omega} + Ay^k = f,\ y^0 \in H,\ k = 0,1,2,...,\ k_1, \tag{10}$$

$$(D + \omega L)C\frac{y^{k+1} - y^k}{\omega}C + ACy^kC = f,\quad y^0 = y^{k_1},\ k = k_1 + 1, k_1 + 2,... . \tag{11}$$

Sufficient convergence conditions of schemes (10)–(11) for the symmetrized Successive OverRelaxation method at any initial approximation are inherited by sufficient convergence conditions of schemes (8)–(9). However, in addition to these restrictions, an additional condition imposed on the iterative parameter is required: ω: $0<\omega < 2$ [5].

3. Complete symmetrization of the Seidel method and the Successive OverRelaxation method. The idea of complete symmetrization methods is close to the methods of ordinary symmetrization. However, when solving problem (7) by the method of complete symmetrization, iterations can start from any corner of a rectangular grid ω_h and calculations are performed in rows or columns (i.e., either from point x_{11} to point $x_{N_1-1N_2-1}$, or from point $x_{N_1-1N_2-1}$ to point x_{11}, or from point x_{1N_2-1} to point x_{N_1-11}, or from point x_{N_1-11}, to point x_{1N_2-1}).

The difference scheme corresponding to the complete symmetrized Seidel method can be represented as:

$$(D + L)(y^{k+1} - y^k) + Ay^k = f_1,\ y^0 \in H,\ k = 0,1,2,...,\ k_1, \tag{12}$$

$$\left(D+L\right)\left(y^{k+1}-y^{k}\right)+Ay^{k}=f_{1},\ y^{0}\in H,\ k=0,1,2,...,\ k_{1}, \tag{13}$$

$$\left(D+L\right)C\left(y^{k+1}-y^{k}\right)+ACy^{k}=f,\ \ y^{0}=y^{k_{1}},\ k=k_{1}+1,k_{1}+2,...\ k_{2}. \tag{14}$$

$$\left(D+L\right)C\left(y^{k+1}-y^{k}\right)C+ACy^{k}C=f,\ \ y^{0}=y^{k_{3}},\ k=k_{3}+1,k_{3}+2,...\ . \tag{15}$$

Sufficient conditions for convergence of schemes (12)–(15) for a complete symmetrized Seidel method are determined from the constraints imposed on the operator A.

The difference scheme corresponding to the Complete Symmetric Successive OverRelaxation method can be represented as:

$$\left(D+\omega L\right)\frac{y^{k+1}-y^{k}}{\omega}+Ay^{k}=f,\ y^{0}\in H,\ k=0,1,2,...,\ k_{1}, \tag{16}$$

$$\left(D+\omega L\right)C\frac{y^{k+1}-y^{k}}{\omega}+ACy^{k}=f,\ \ y^{0}=y^{k_{1}},\ k=k_{1}+1,k_{1}+2,...\ k_{2}. \tag{17}$$

$$\left(D+\omega L\right)\frac{y^{k+1}-y^{k}}{\omega}C+Ay^{k}C=f,\ \ y^{0}=y^{k_{2}},\ k=k_{2}+1,k_{2}+2,...\ k_{3}. \tag{18}$$

$$\left(D+\omega L\right)C\frac{y^{k+1}-y^{k}}{\omega}C+ACy^{k}C=f,\ \ y^{0}=y^{k_{3}},\ k=k_{3}+1,k_{3}+2,...\ . \tag{19}$$

Sufficient convergence conditions of the Complete Symmetric Successive OverRelaxation method coincide with sufficient convergence conditions of the complete symmetrized Seidel method, and a restriction on the iterative parameter is added: ω: $0 < \omega < 2$.

Results. We illustrate the calculation results using the described methods on a grid ω_h at $N = N_1 = N_2$.

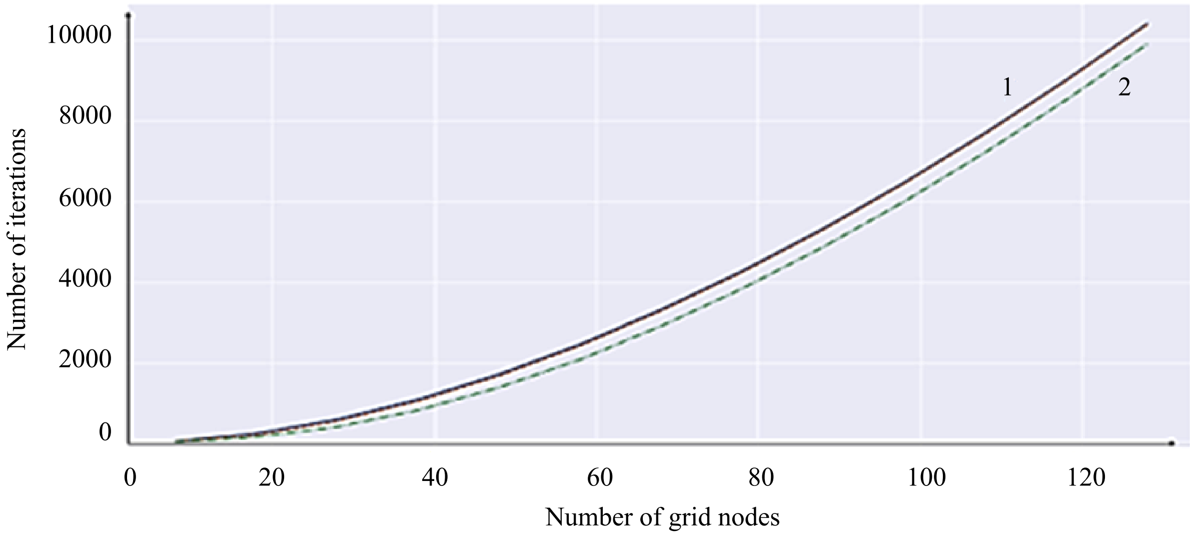

Fig. 1. Graph of the dependence of the number of iterations on the number of grid nodes when solving the problem using: 1 — the Seidel method and the symmetrized Seidel method (the lines coincide); 2 — the complete symmetrized Seidel method

The problem is solved:

$$\begin{cases}\sum_{\alpha=1}^{2} y_{\bar{x}_\alpha x_\alpha}=f\left(x\right), & x\in\omega_{h},\\ y\left(x\right)=0, & x\in\gamma.\end{cases} \tag{20}$$

The function of the right part $f(x)$ was chosen in such a way that $y(x) = x_1(x–x_1)x_2(x–x_2)$ is an exact solution to the problem (20).

For the Symmetric Successive OverRelaxation method and the Symmetric Successive OverRelaxation method, the iterative parameter ω was chosen according to the formula [5]:

$$\omega=\frac{2}{1+\sin\left(\pi/n\right)}.$$

The calculations are performed until the accuracy $\varepsilon = 10^{-4}$ is reached, where $\varepsilon = \frac{\left\|r^{N(\varepsilon)}\right\|_C}{\left\|r^0\right\|_C}$, $\left\|r^{N(\varepsilon)}\right\|_C$ is the grid norm C of the discrepancy at the final iteration, at which the specified accuracy is achieved, $\left\|r^0\right\|_C$ is the norm from the initial discrepancy.

Figures 1 and 2 show the results of calculations related to the solution of problem (20) using the considered iterative methods. The dependence of the number of grid nodes on the number of iterations required to achieve the required accuracy is demonstrated ε.

In accordance with the graphs (Fig. 1), a slight decrease in the required number of iterations for the symmetrized Seidel method should be noted. The complete Symmetric Successive OverRelaxation method requires significantly fewer iterations compared to its unsymmetrized counterpart. In terms of the costs of arithmetic operations per iteration, the basic methods and their symmetrized analogues differ slightly.

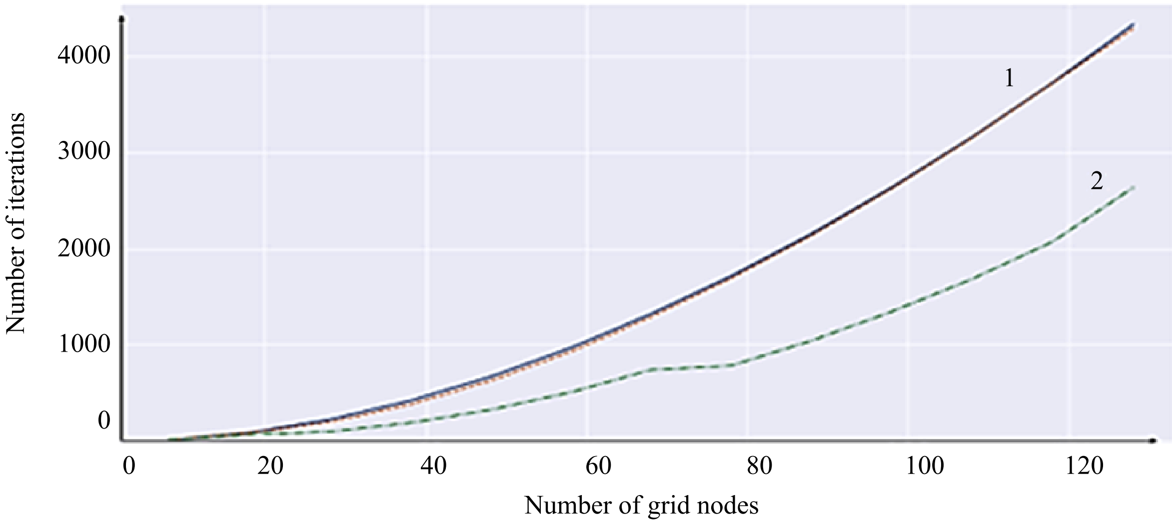

Fig. 2. Graph of the dependence of the number of iterations on the number of grid nodes when solving the problem using: 1 — the Successive OverRelaxation and the Symmetric Successive OverRelaxation method (the lines coincide); 2 — the complete Symmetric Successive OverRelaxation method

This is confirmed by a comparative analysis of the data obtained, shown in Table 1.

Table 1

Calculation results using various iterative methods

Iterative method	$N = 32$	$N = 64$	$N = 128$
Seidel method	757	2947	10420
Symmetrized Seidel method	714	2914	10400
Complete symmetrization of the Seidel method	538	2550	9914
Successive OverRelaxation method	281	1131	4335
Symmetric Successive OverRelaxation method	253	1111	4293
Complete Symmetric Successive OverRelaxation method	101	521	2637

Discussion and Conclusions. The article proposes methods of symmetrization for the Seidel and the Successive OverRelaxation methods. The use of a complete Symmetric Successive OverRelaxation method can significantly reduce the number of iterations required to achieve a given accuracy. It helps to halve the required number of iterations without additional computational costs. This work has practical significance. The developed software tool makes it possible to use it to solve specific physical problems, including as an element of a software package [6–9].

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