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Mathematical Model of Three-Component Suspension Transport

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Abstract
Introduction. Prediction of suspension deposition zones is required to assess and minimize the negative impact on the ecosystem of the reservoir during dredging within the framework of large-scale engineering projects, prediction of suspension deposition zones is required to assess and minimize the negative impact on the ecosystem of the reservoir. It is necessary to build a mathematical model that takes into account many factors that have a significant impact on the accuracy of forecasts. The aim of the work is to construct a mathematical model of transport of multicomponent suspension, taking into account the composition of the soil (different diameter of the suspension particles), the flow velocity of the water flow, the complex geometry of the coastline and bottom, wind stresses and friction on the bottom, turbulent exchange, etc.
Materials and Methods. A mathematical model for the transport of a multicomponent suspension and an approximation of the proposed continuous model with the second order of accuracy with respect to the steps of the spatial grid are described, considering the boundary conditions of the Neumann and Robin type. The approximation of the hydrodynamics model is obtained based on splitting schemes by physical processes, which guarantee fulfillment mass conservation for discrete model.
Results. The proposed mathematical model formed the basis of the developed software package that allows to simulate the process of sedimentation of a multicomponent suspension. The results of the work of the software package on the model problem of sedimentation of a three-component suspension in the process of soil dumping during dredging are presented.
Discussions and Conclusions. The mathematical model of transport of three-component suspension is described and software implemented. The developed software allows to simulate the process of deposition of suspended particles of various diameters on the bottom, and to evaluate its effect on the bottom topography and changes in the bottom composition. The developed software package also allows to analyze the process of sediment movement in the case of resuspension of multicomponent bottom sediments of the reservoir, which causes secondary pollution of the reservoir.

Keywords: suspension transport, multicomponent suspension, three-dimensional hydrodynamics model, splitting schemes, numerical methods

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Научная статья

Математическая модель транспорта трехкомпонентной взвеси

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Аннотация
Введение. При проведении дноуглубительных работ в рамках реализации масштабных инженерных проектов требуется прогнозирование зон осаждения взвеси для оценки и минимизации негативного влияния на экосистему водоема. Для решения подобных задач необходимо построение математической модели, учитывающей множество факторов, оказывающих существенное влияние на точность прогнозов. Целью работы является построение математической модели транспорта многокомпонентной взвеси, учитывающей состав грунта (различный диаметр частиц взвеси), скорость течения водного потока, сложную геометрию береговой линии и дна, ветровые напряжения и трение о дно, турбулентный обмен и др.
Материалы и методы. Описана математическая модель транспорта многокомпонентной взвеси и аппроксимация предложенной непрерывной модели со вторым порядком точности относительно шагов пространственной сетки с учетом граничных условий второго и третьего рода. Аппроксимация модели гидродинамики представлена на основе схем расщепления по физическим процессам, которая обеспечивает выполнение закона сохранения массы в разностной схеме.
Результаты исследования. Предлагаемая математическая модель легла в основу разработанного программного комплекса, позволяющего моделировать процесс осаждения многокомпонентной взвеси. Приведены результаты работы программного комплекса на модельной задаче осаждения трехкомпонентной взвеси в процессе дампинга грунта при проведении дноуглубительных работ.
Обсуждения и заключения. Описана программная математическая модель транспорта трехкомпонентной взвеси. Разработанный программный комплекс позволяет моделировать процесс осаждения взвешенных частиц различного диаметра на дно и оценивать его влияние на рельеф и изменение состава дна. Разработанный программный комплекс также позволяет анализировать процесс движения наносов в случае взмучивания многокомпонентных донных отложений водоема, вызывающий вторичное загрязнение водоема.

Ключевые слова: транспорт взвеси, многокомпонентная взвесь, трехмерная модель гидродинамики, схемы расщепления, численные методы

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Introduction. The implementation of large-scale engineering projects, such as the construction of bridges, the expansion of the water area accessible to navigation, requires work that has a significant impact on both the bottom relief and the ecosystem of the reservoir as a whole. For example, during dredging, a significant amount of suspension enters the water, which in the process of settling to the bottom or secondary agitation can negatively affect the productive and destructive processes of the aquatic ecosystem [1–2]. To assess the possible damage caused to the ecosystem during the dumping of soil during dredging, it is necessary to pre-assess the areas of the water area in which the suspension will settle and in which its agitation is possible, which leads to secondary pollution of the water body. To predict the deposition zones of suspended particles, a mathematical model of suspension transport is proposed based on a system of initial boundary value problems, including the calculation of hydrodynamic characteristics of the water area and suspension transport.

We describe an approach to the approximation of a continuous model with a second order of accuracy with respect to the steps of the spatial grid , taking into account the boundary conditions of the second and third kind for the proposed three — dimensional model of multicomponent suspension transport . The proposed mathematical model of the transport of suspended particles is supplemented by a three-dimensional model of hydrodynamics, which allows calculating the fields of the velocity vector of the water flow [3–4]. The proposed mathematical model formed the basis of the developed software package that allows modeling the deposition process of multicomponent suspension. The results of the work of the software package on the model problem of deposition of a three-component suspension in the process of dumping soil during dredging are presented.

Materials and methods

1. Problem statement. To construct a mathematical model of multicomponent suspension transport, we use the diffusion-convection equation, which can be written in the following form [3]:

$$(c_r)_t' + (uc_r)_x' + (vc_r)_y' + ((w+w_{s,r})\,c_r)_z' = \left(\mu (c_r)_x'\right)_x' + \left(\mu (c_r)_y'\right)_y' + \left(\nu (c_r)_z'\right)_z' + F_r\,, \tag{1}$$

where c_r is the concentration of the r-th fraction of the suspension, mg/l; $\mathbf{V}=\{u,v,w\}$ are the components of the velocity vector of the water flow, m/s; $w_{s,r}$ is the deposition rate of the r-th fraction of the suspension, m/s; μ, ν are the horizontal and vertical components of the turbulent exchange coefficient, respectively, m^2/s; F_r is the function describing the intensity of the distribution of sources of the rth fraction of the suspension, mg/ (l·s).

Equation (1) is considered under the following initial $c_r(x, y, z, 0) = c_{0r}(x, y, z)$ and boundary conditions:

– on the free surface: $(c^n)_z' = 0$;

– near the bottom surface: $\nu (c_r)_z' = -w_{s,r} c_r$;

– on the side surface: $(c_r)_\mathbf{n}' = 0$, if $(\mathbf{V},\mathbf{n}) \geq 0$, and $\mu (c_r)_\mathbf{n}' = (\mathbf{V},\mathbf{n}) c_r$, if $(\mathbf{V},\mathbf{n}) < 0$, where $(\mathbf{V},\mathbf{n})$ is the normal component of the velocity vector, **n** is the normal vector directed inside the computational domain.

The diffusion-convection equation (1) is supplemented with a three-dimensional model of the hydrodynamics of shallow water bodies [5] to calculate the velocity vector of the water flow:

– equations of motion (Navier-Stokes):

$$\begin{aligned} u_t' + uu_x' + vu_y' + wu_z' &= -P_x'/\rho + (\mu u_x')_x' + (\mu u_y')_y' + (\nu u_z')_z', \\ v_t' + uv_x' + vv_y' + wv_z' &= -P_y'/\rho + (\mu v_x')_x' + (\mu v_y')_y' + (\nu v_z')_z', \\ w_t' + uw_x' + vw_y' + ww_z' &= -P_z'/\rho + (\mu w_x')_x' + (\mu w_y')_y' + (\nu w_z')_z' + g, \end{aligned} \tag{2}$$

– the continuity equation in the case of variable density:

$$\rho_t' + (\rho u)_x' + (\rho v)_y' + (\rho w)_z' = 0. \tag{3}$$

where P is the pressure, Pa; ρ is the density, kg/m^3; g is the acceleration of gravity, m/s^2.

The system of equations (2)–(3) is considered under the following boundary conditions:

– the entrance $u = u_0$, $v = v_0$, $P_\mathbf{n}' = 0$, $V_\mathbf{n}' = 0$;

– lateral border (shore and bottom) $\rho\mu u_\mathbf{n}' = -\tau_x$, $\rho\mu v_\mathbf{n}' = -\tau_y$, $V_\mathbf{n} = 0$, $P_\mathbf{n}' = 0$;

– upper bound $\rho\mu u_\mathbf{n}' = -\tau_x$, $\rho\mu v_\mathbf{n}' = -\tau_y$, $w = -\omega - P_t'/(\rho g)$, $P_\mathbf{n}' = 0$,

where ω is the intensity of liquid evaporation, τ_x, τ_y are the components of the tangential stress.

2. Approximation of the suspended particles transport problem. Let us consider an approximation of the three-dimensional problem of transport of a one-component suspension based on the expression (1) (for each individual fraction, the equation is written similarly):

$$c_t' + (uc)_x' + (vc)_y' + (wc)_z' = \left(\mu c_x'\right)_x' + \left(\mu c_y'\right)_y' + \left(\nu c_z'\right)_z', \tag{4}$$

where the velocity component w implicitly takes into account the deposition rate of the suspension fraction in question w_s.

We introduce a uniform grid in time $\overline{\omega}_\tau = \{t^n = n\tau;\ n = \overline{0, N_t};\ N_t\tau = T\}$, where τ is the time step, N_t is the number of time layers, T is the duration of the modeling interval.

Suppose that the calculated area is inscribed in a parallelepiped $G = \{0 < x < L_x, 0 < y < L_y, 0 < z < L_z\}$ we obtain the closure of the area G by joining the faces of the parallelepiped, that is, we define as $\overline{G} = \{0 \le x \le L_x, 0 \le y \le L_y, 0 \le z \le L_z\}$.

Thus, we come to the chain of initial boundary value problems:

$$(c^n)_t' + \operatorname{div}(\mathbf{V} \cdot c^n) = \operatorname{div}(\mathbf{k} \cdot \operatorname{grad}\ c^n), \tag{5}$$

where $\mathbf{k} = \{\mu, \mu, \nu\}$ is the coefficient of turbulent exchange, $(x, y, z, t) \in G \times [0 < t \le T]$, $t_{n-1} < t \le t_n$, $cn\,(x, y, z, t_{n-1}) = c^{n-1}$ (x, y, z, t_{n-1}), (x, y, z, t_{n-1}), $(x, y, z) \in G$.

In this case, the initial and boundary conditions will be written as:

– initial condition $c\,(x, y, z, 0) = c_0\,(x, y, z)$, $(x, y, z) \in \overline{G}$;

– boundary condition on a free surface: $(c_r)'_z = 0$; , $(x, y, z) \in \Sigma_0$;
– boundary condition at the bottom: $\nu(c^n)'_z = -w_s c^n$, $(x, y, z) \in \Sigma_H$;
– boundary condition on the lateral surface: $(c^n)'_{\mathbf{n}} = 0$, if $(\mathbf{V}, \mathbf{n}) \geq 0$ and $\mu (c_r)'_{\mathbf{n}} = (\mathbf{V}, \mathbf{n}) c_r$, , if $(\mathbf{V}, \mathbf{n}) < 0$,
where $\Sigma_0 = \{0 \leq x \leq L_x, 0 \leq y \leq L_y, z = 0\}$ is the upper face of the parallelepiped $\overline{G}$, $\Sigma_H = \{0 \leq x \leq L_x, 0 \leq y \leq L_y, z = L_z\}$ is the lower face of the parallelepiped $\overline{G}$.

Let's write down a term describing the convective transfer of suspended substances in a symmetrical form. Such an approach to the discretization of the problem will allow us to construct a difference operator with the property of skew symmetry [6]:

$$\operatorname{div}(\mathbf{V} \cdot c^n) = \frac{1}{2}\left[u\frac{\partial c^n}{\partial x} + v\frac{\partial c^n}{\partial y} + w\frac{\partial c^n}{\partial z} + \frac{\partial}{\partial x}(uc^n) + \frac{\partial}{\partial y}(vc^n) + \frac{\partial}{\partial z}(wc^n) \right]. \tag{6}$$

To approximate problem (5) , we introduce a uniform computational grid $\overline{\omega} = \overline{\omega}_x \times \overline{\omega}_y \times \overline{\omega}_z$, where:

$$\overline{\omega}_x = \{x_i : x_i = ih_x;\ i = \overline{0, N_x};\ h_x N_x = L_x\},$$
$$\overline{\omega}_y = \{y_j : y_j = jh_y;\ j = \overline{0, N_y};\ h_y N_y = L_y\},$$
$$\overline{\omega}_z = \{z_k : z_k = kh_z;\ k = \overline{0, N_z};\ h_z N_z = L_z\},$$

where h_x, h_y, h_z are the steps in the spatial coordinate directions O_x, O_y and O_z , respectively, N_x, N_y, N_z , are the numbers of nodes of the computational grid in each of the spatial directions, L_x, L_y, L_z are the lengths of the computational intervals in each of the spatial directions..

The set of internal nodes of the computational grid is denoted as ω_x, ω_y, ω_z.

The approximation of problem (5) on a space-time grid $\overline{\omega}_\tau \times \overline{\omega}$ is performed by specifying the velocities (convective transfer coefficients) at the nodes of the grids shifted by half a step along the coordinate directions O_x and O_y.

For the convective transport operator given in symmetric form (6) in equation (5), we have:

$$\begin{aligned} Cc^n \equiv{}& \frac{1}{2h_x}\left(u^n(x+0.5h_x, y, z)\cdot \overline{c}^n(x+h_x, y, z) - u^n(x-0.5h_x, y, z)\cdot \overline{c}^n(x-h_x, y, z)\right) + \\ &+ \frac{1}{2h_y}\left(v^n(x, y+0.5h_y, z)\cdot \overline{c}^n(x, y+h_y, z) - v^n(x, y-0.5h_y, z)\cdot \overline{c}^n(x, y-h_y, z)\right) + \\ &+ \frac{1}{2h_z}\left(w^n(x, y, z+0.5h_z)\cdot \overline{c}^n(x, y, z+h_z) - w^n(x, y, z-0.5h_z)\cdot \overline{c}^n(x, y, z-h_z)\right), \end{aligned} \tag{7}$$

where $\overline{c}^n$ denotes a grid function $\overline{c}^n \equiv c(x, y, z, t_n)$, $(x, y, z) \in \omega$, and through c^n denotes a sufficiently smooth function of continuous variables (x, y, z, t).

For the diffusion transfer operator in equation (5) we have:

$$\begin{aligned} Dc^n \equiv{}& \frac{1}{h_x}\left(\mu(x+0.5h_x, y, z)\frac{\overline{c}^n(x+h_x, y, z) - \overline{c}^n(x, y, z)}{h_x} - \right. \\ &\left. - \mu(x-0.5h_x, y, z)\frac{\overline{c}^n(x, y, z) - \overline{c}^n(x-h_x, y, z)}{h_x} \right) + \frac{1}{h_y}\left(\mu(x, y+0.5h_y, z)\frac{\overline{c}^n(x, y+h_y, z) - \overline{c}^n(x, y, z)}{h_y} - \right. \\ &\left. - \mu(x, y-0.5h_y, z)\frac{\overline{c}^n(x, y, z) - \overline{c}^n(x, y-h_y, z)}{h_y} \right) + \frac{1}{h_z}\left(\nu(x, y, z+0.5h_z)\frac{\overline{c}^n(x, y, z+h_z) - \overline{c}^n(x, y, z)}{h_z} - \right. \\ &\left. - \nu(x, y, z-0.5h_z)\cdot\frac{\overline{c}^n(x, y, z) - \overline{c}^n(x, y, z-h_z)}{h_z} \right). \end{aligned} \tag{8}$$

Taking into account the recorded approximations (7)–(8), we obtain the following type of approximation of equation (5) in the inner nodes of the grid:

$$\frac{\bar{c}^n - \bar{c}^{n-1}}{\tau} + \frac{1}{2h_x}\left(u^n\left(x+0.5h_x,y,z\right)\bar{c}^n\left(x+h_x,y,z\right) - u^n\left(x-0.5h_x,y,z\right)\bar{c}^n\left(x-h_x,y,z\right)\right) +$$
$$+\frac{1}{2h_y}\left(v^n\left(x,y+0.5h_y,z\right)\bar{c}^n\left(x,y+h_y,z\right) - v^n\left(x,y-0.5h_y,z\right)\bar{c}^n\left(x,y-h_y,z\right)\right) +$$
$$+\frac{1}{2h_z}\left(w^n\left(x,y,z+0.5h_z\right)\bar{c}^n\left(x,y,z+h_z\right) - w^n\left(x,y,z-0.5h_z\right)\bar{c}^n\left(x,y,z-h_z\right)\right) = \quad (9)$$
$$=\frac{1}{h_x}\left(\mu\left(\mathrm{x}+0.5h_x,y,z\right)\frac{\bar{c}^n\left(x+h_x,x,z\right)-\bar{c}^n\left(x,y,z\right)}{h_x} - \mu\left(x-0.5h_x,y,z\right)\frac{\bar{c}^n\left(x,y,z\right)-\bar{c}^n\left(x-h_x,y,z\right)}{h_x}\right)+$$
$$+\frac{1}{h_y}\left(\mu\left(x,y+0.5h_y,z\right)\frac{\bar{c}^n\left(x,y,z\right)-\bar{c}^n\left(x,y,z\right)}{h_y} - \mu\left(x,y-0.5h_y,z\right)\frac{\bar{c}^n\left(x,y,z\right)-\bar{c}^n\left(x,y-h_y,z\right)}{h_y}\right)+$$
$$+\frac{1}{h_z}\left(\nu\left(x,y,z+0.5h_z\right)\cdot\frac{\bar{c}^n\left(x,y,z+h_z\right)-\bar{c}^n\left(x,y,z\right)}{h_z} - \nu\left(x,y,z-0.5h_z\right)\frac{\bar{c}^n\left(x,y,z\right)-\bar{c}^n\left(x,y,z-h_z\right)}{h_z}\right).$$

We supplement the obtained approximation (9) with initial and boundary conditions. To set boundary conditions on the bottom, free and lateral surfaces of the considered area it is convenient to introduce an expanded grid [7] $\bar{\omega}^* = \bar{\omega}_x^* \times \bar{\omega}_y^* \times \bar{\omega}_z^*$, where

$$\bar{\omega}_x^* = \left\{x_i : x_i = ih_x;\ i = \overline{-1, N_x+1};\ h_x N_x = L_x\right\},$$
$$\bar{\omega}_y^* = \left\{y_j : y_j = jh_y;\ j = \overline{-1, N_y+1};\ h_y N_y = L_y\right\},$$
$$\bar{\omega}_z^* = \left\{z_k : z_k = kh_z;\ k = \overline{-1, N_z+1};\ h_z N_z = L_z\right\}\cdot$$

In the future, we will assume that:

$$\bar{c}^n(x,y,z) = 0, \quad (10)$$

where $\bar{\omega}^* \setminus \bar{\omega}$ are the boundary nodes of the grid $\bar{\omega}^*$.

We also consider the values of the components of the velocity vector of the aqueous medium in the grid nodes $\bar{\omega}^* \setminus \bar{\omega}$ with fractional index values to be known. For grid nodes $\bar{\omega}^* \setminus \bar{\omega}$, that are located outside the reservoir, the values of the velocity vector components are set to zero.

In the case of flows on the lateral faces of the region G, coinciding in the direction with the external normals to the faces (case $(\mathbf{V}, \mathbf{n}) \geq 0$), the Neumann boundary conditions take place. This case can be written as:

$$\begin{gathered} u^n(0.5h_x,y,z)+u^n(-0.5h_x,y,z)<0, \\ u^n(L_x-0.5h_x,y,z)+u^n(L_x+0.5h_x,y,z)>0, \\ v^n(x,0.5h_y,z)+v^n(x,-0.5h_y,z)<0, \\ v^n(x,0.5h_y,z)+v^n(x,-0.5h_y,z)<0, \end{gathered} \quad (11)$$

We write down an approximation of the boundary conditions of the second kind for the convective transport operator. Consider the case $x = 0$, $0 < y < L_y$, $0 < z < L_z$. In this case, the expression can be considered as a difference approximation of the convective term:

$$\frac{1}{2h_x}\left(u^n\left(0.5h_x,y,z\right)\bar{c}^n\left(h_x,y,z\right) - u^n\left(-0.5h_x,y,z\right)\bar{c}^n\left(-h_x,y,z\right)\right)$$

Expression (12) approximates the convective term with an error $O(h_x^2)$. In addition to the form (12), the approximation of the convective term with an error $O(h_x^2)$ can be written as:

$$\frac{\bar{c}^n(h_x,y,z) - \bar{c}^n(-h_x,y,z)}{2h_x} = 0,$$

from where we get:

$$\bar{c}^n(-h_x, y, z) = \bar{c}^n(h_x, y, z). \tag{13}$$

From approximations (12) and (13) we obtain:

$$C_x\left(c^n\right)\Big|_{x=0} \equiv \frac{1}{2h_x}\bar{c}^n(h_x, y, z)\left(u^n(0.5h_x, y, z) - u^n(-0.5h_x, y, z)\right). \tag{14}$$

Similarly, the cases $x = L_x$, $y = 0$, $y = Ly$, $z = 0$ (boundary condition on a free surface) are written. We get:

$$C_x\left(c^n\right)\Big|_{x=L_x} \equiv \frac{1}{2h_x}\cdot\bar{c}^n(L_x - h_x, y, z)\cdot\left(u^n(L_x + 0.5h_x, y, z) - u^n(L_x - 0.5h_x, y, z)\right), \tag{15}$$

$$C_y\left(c^n\right)\Big|_{y=0} \equiv \frac{1}{2h_y}\cdot\bar{c}^n\left(x, h_y, z\right)\cdot\left(v^n\left(x, 0.5h_y, z\right) - v^n\left(x, -0.5h_y, z\right)\right), \tag{16}$$

$$C_y\left(c^n\right)\Big|_{y=L_y} \equiv \frac{1}{2h_y}\cdot\bar{c}^n\left(x, L_y - h_y, z\right)\cdot\left(v^n\left(x, L_y + 0.5h_y, z\right) - v^n\left(x, L_y - 0.5h_y, z\right)\right), \tag{17}$$

$$C_z\left(c^n\right)\Big|_{z=0} \equiv \frac{1}{2h_z}\cdot\bar{c}^n\left(x, y, h_z\right)\cdot\left(w^n\left(x, y, 0.5h_z\right) - w^n\left(x, y, -0.5h_z\right)\right). \tag{18}$$

We write down an approximation of the boundary conditions of the second kind for the diffusion transfer operator. Consider the case $x = 0$, $0 < y < L_y$, $0 < z < L_z$. In this case, the expression can be considered as a difference approximation of the diffusion term on an extended grid:

$$D_x\left(c^n\right)\Big|_{x=0} \equiv \frac{1}{h_x}\left(\mu(0.5h_x, y, z)\frac{\bar{c}^n(h_x, y, z) - \bar{c}^n(0, y, z)}{h_x} - \right.$$
$$\left. - \mu(-0.5h_x, y, z)\cdot\frac{\bar{c}^n(0, y, z) - \bar{c}^n(-h_x, y, z)}{h_x}\right). \tag{19}$$

When the first condition of (11) is fulfilled, we obtain that $\bar{c}^n(-h_x, y, z) = \bar{c}^n(h_x, y, z)$. Then, taking into account the last equality and expression (19), we obtain the following approximation of the diffusion transfer operator in the case: $x = 0$, $0 < y < L_y$, $0 < z < L_z$:

$$D_x\left(c^n\right)\Big|_{x=0} \equiv \frac{1}{h_x^2}\left(\left(\mu(0.5h_x, y, z) + \mu(-0.5h_x, y, z)\right)\left(\bar{c}^n(h_x, y, z) - \bar{c}^n(0, y, z)\right)\right). \tag{20}$$

Similarly, the cases $x = L_x$, $y = 0$, $y = L_y$ are written. For example, in the case $x = L_x$, $0 < y < L_y$, $0 < z < L_z$, when the second condition from (11) is met and taking into account equality $\bar{c}^n(L_x + h_x, y, z) = \bar{c}^n(L_x - h_x, y, z)$, we get:

$$D_x\left(c^n\right)\Big|_{x=L_x} \equiv \frac{1}{h_x^2}\left(\left(\mu(L_x + 0.5h_x, y, z) + \mu(L_x - 0.5h_x, y, z)\right)\left(\bar{c}^n(L_x - h_x, y, z) - \bar{c}^n(L_x, y, z)\right)\right). \tag{21}$$

Similarly for $y = 0$ and $y = L_y$:

$$D_y\left(c^n\right)\Big|_{y=0} \equiv \frac{1}{h_y^2}\left(\left(\mu\left(x, 0.5h_y, z\right) + \mu\left(x, -0.5h_y, z\right)\right)\left(\bar{c}^n\left(x, h_y, z\right) - \bar{c}^n\left(x, 0, z\right)\right)\right), \tag{22}$$

$$D_y\left(c^n\right)\Big|_{y=L_y} \equiv \frac{1}{h_y^2}\left(\left(\mu\left(x, L_y + 0.5h_y, z\right) + \mu\left(x, L_y - 0.5h_y, z\right)\right)\left(\bar{c}^n\left(x, L_y - h_y, z\right) - \bar{c}^n\left(x, L_y, z\right)\right)\right). \tag{23}$$

For a free surface (case $y = 0$) taking into account $\nu(x, y, -0{,}5h_z) = 0$ the approximation of the diffusion operator is written as:

$$D_z\left(c^n\right)\Big|_{z=0} \equiv \frac{1}{h_z^2}\nu\left(x, y, 0.5h_z\right)\left(\bar{c}^n\left(x, y, h_z\right) - \bar{c}^n\left(x, y, 0\right)\right). \tag{24}$$

Consider the approximation of the diffusion transfer operator on the bottom surface ($z = L_z$) on an extended grid. Formally , the approximation of the diffusion term can be written as:

$$D_z\left(c^n\right)\Big|_{z=L_z} \equiv \frac{1}{h_z}\left(\nu\left(x, y, L_z + 0.5h_z\right)\frac{\left(\bar{c}^n\left(x, y, L_z + h_z\right) - \bar{c}^n\left(x, y, L_z\right)\right)}{h_z} - \right. \tag{25}$$

$$-\nu\left(x,y,L_z-0.5h_z\right)\frac{\left(\bar{c}^n\left(x,y,L_z\right)-\bar{c}^n\left(x,y,L_z-h_z\right)\right)}{h_z}\Bigg).$$

Let's use a second-order approximation of the accuracy $O(h^2_z)$ of the boundary condition of the third kind $\nu\left(c^n\right)'_z=-w_s c^n$, the relation:

$$\nu\left(x,y,L_z\right)\frac{\left(\bar{c}^n\left(x,y,L_z+h_z\right)-\bar{c}^n\left(x,y,L_z-h_z\right)\right)}{2h_z}=-w_s\bar{c}^n\left(x,y,L_z\right). \quad (26)$$

Let 's use the ratio:

$$\nu\left(x,y,L_z\right)=\frac{1}{2}\left(\nu\left(x,y,L_z+0.5h_z\right)+\nu\left(x,y,L_z-0.5h_z\right)\right)+O\left(h_z^2\right). \quad (27)$$

Substituting (27) into (26), we get:

$$\left(\nu\left(x,y,L_z+0.5h_z\right)+\nu\left(x,y,L_z-0.5h_z\right)\right)\frac{\left(\bar{c}^n\left(x,y,L_z+h_z\right)-\bar{c}^n\left(x,y,L_z-h_z\right)\right)}{4h_z}=-w_s\bar{c}^n\left(x,y,L_z\right). \quad (28)$$

From equation (28) we have:

$$\bar{c}^n\left(x,y,L_z+h_z\right)=-\frac{4w_s h_z}{\left(\nu\left(x,y,L_z+0.5h_z\right)+\nu\left(x,y,L_z-0.5h_z\right)\right)}\bar{c}^n\left(x,y,L_z\right)+\bar{c}^n\left(x,y,L_z-h_z\right). \quad (29)$$

Then the approximation of the diffusion term in the diffusion-convection equation of the suspension has the form:

$$\begin{aligned}D_z\left(c^n\right)\Big|_{z=L_z}\equiv\frac{1}{h_z}\Bigg(&\frac{4w_s\nu\left(x,y,L_z+0.5h_z\right)}{\nu\left(x,y,L_z+0.5h_z\right)+\nu\left(x,y,L_z-0.5h_z\right)}\bar{c}^n\left(x,y,L_z\right)+\\&+\frac{1}{h_z}\nu\left(x,y,L_z+0.5h_z\right)\left(\bar{c}^n\left(x,y,L_z-h_z\right)-\bar{c}^n\left(x,y,L_z\right)\right)-\\&-\nu\left(x,y,L_z-0.5h_z\right)\frac{\left(\bar{c}^n\left(x,y,L_z\right)-\bar{c}^n\left(x,y,L_z-h_z\right)\right)}{h_z}\Bigg).\end{aligned} \quad (30)$$

In a similar way , it is possible to obtain an approximation of the convective operator for $z = L_z$:

$$\begin{aligned}C_z\left(c^n\right)\Big|_{z=L_z}\equiv\frac{1}{2h_z}\Bigg(&\frac{4w_s h_z w^n\left(x,y,z+0.5h_z\right)}{\nu\left(x,y,L_z+0.5h_z\right)+\nu\left(x,y,L_z-0.5h_z\right)}\bar{c}^n\left(x,y,L_z\right)+\\&+\left(w^n\left(x,y,L_z+0.5h_z\right)-w^n\left(x,y,L_z-0.5h_z\right)\right)\bar{c}^n\left(x,y,L_z-h_z\right)\Bigg).\end{aligned} \quad (31)$$

The obtained approximations of the diffusion (30) and convective (31) transfer operators at the boundary nodes (at $z = L_z$) are suitable for bottoms with different morphological characteristics ("liquid" bottom, impermeable bottom, etc.) when the turbulent exchange coefficient ν is set accordingly.

After constructing the scheme, it is necessary to investigate the monotonicity, stability and convergence of the difference scheme. The study of these properties uses physically motivated constraints of the Peclet grid number and the Courant number and is based on the maximum grid principle and, due to the limited scope of the article, is not given here.

3. Approximation of three-dimensional hydrodynamics model. To approximate the model (2)–(3), we will carry out on the calculated grid $\bar{\omega}=\bar{\omega}_\tau\times\bar{\omega}_h$. To approximate the model (2)–(3), we use splitting schemes for physical processes [8]. According to this method, the initial model of hydrodynamics (2)–(3) will be divided into three subtasks [6, 9].

The first subtask is represented by the diffusion-convection equation, on the basis of which the components of the field of the velocity vector of the water flow on the intermediate time layer are calculated:

$$\frac{\tilde{u}-u}{\tau}+u\bar{u}_x'+v\bar{u}_y'+w\bar{u}_z'=\left(\mu\bar{u}_x'\right)_x'+\left(\mu\bar{u}_y'\right)_y'+\left(\nu\bar{u}_z'\right)_z',$$

$$\frac{\tilde{v}-v}{\tau}+u\bar{v}_x'+v\bar{v}_y'+w\bar{v}_z'=\left(\mu\bar{v}_x'\right)_x'+\left(\mu\bar{v}_y'\right)_y'+\left(\nu\bar{v}_z'\right)_z',$$

$$\frac{\tilde{w}-w}{\tau}+u\bar{w}_x'+v\bar{w}_y'+w\bar{w}_z'=\left(\mu\bar{w}_x'\right)_x'+\left(\mu\bar{w}_y'\right)_y'+\left(\nu\bar{w}_z'\right)_z'+g\left(\frac{\rho_0}{\rho}-1\right),$$

where u, v, w are the components of the velocity vector on the previous time layer; $\tilde{u}, \tilde{v}, \tilde{w}$ are the components of the velocity vector on the intermediate time layer; $\bar{u} = \sigma\tilde{u} + (1-\sigma)u$, $\sigma \in [0,1]$ is the weighting factor or the weight of the scheme.

Based on the second subtask, the pressure field is calculated:

$$P''_{xx} + P''_{yy} + P''_{zz} = \frac{\hat{\rho} - \rho}{\tau^2} + \frac{(\hat{\rho}\tilde{u})'_x}{\tau} + \frac{(\hat{\rho}\tilde{v})'_y}{\tau} + \frac{(\hat{\rho}\tilde{w})'_z}{\tau}.$$

Based on the third subtask, the components of the field of the velocity vector of the water flow on the next time layer are calculated using explicit formulas:

$$\frac{\hat{u} - \tilde{u}}{\tau} = -\frac{1}{\hat{\rho}} P'_x, \quad \frac{\hat{v} - \tilde{v}}{\tau} = -\frac{1}{\hat{\rho}} P'_y, \quad \frac{\hat{w} - \tilde{w}}{\tau} = -\frac{1}{\rho} P'_z,$$

where $\hat{u}, \hat{v}, \hat{w}$ are the components of the velocity vector on the current time layer.

The approximation of the problem of calculating the velocity field of the water medium by spatial variables is performed on the basis of the balance method.

Results. Based on the presented mathematical model of multicomponent suspension transport, a software package in *C*++ has been developed that takes into account various factors that affect the accuracy of the forecasts obtained, among which one can distinguish the complex geometry of the bottom and coastline, wind currents and friction on the bottom, the presence of a significant gradient in the density of the aquatic environment.

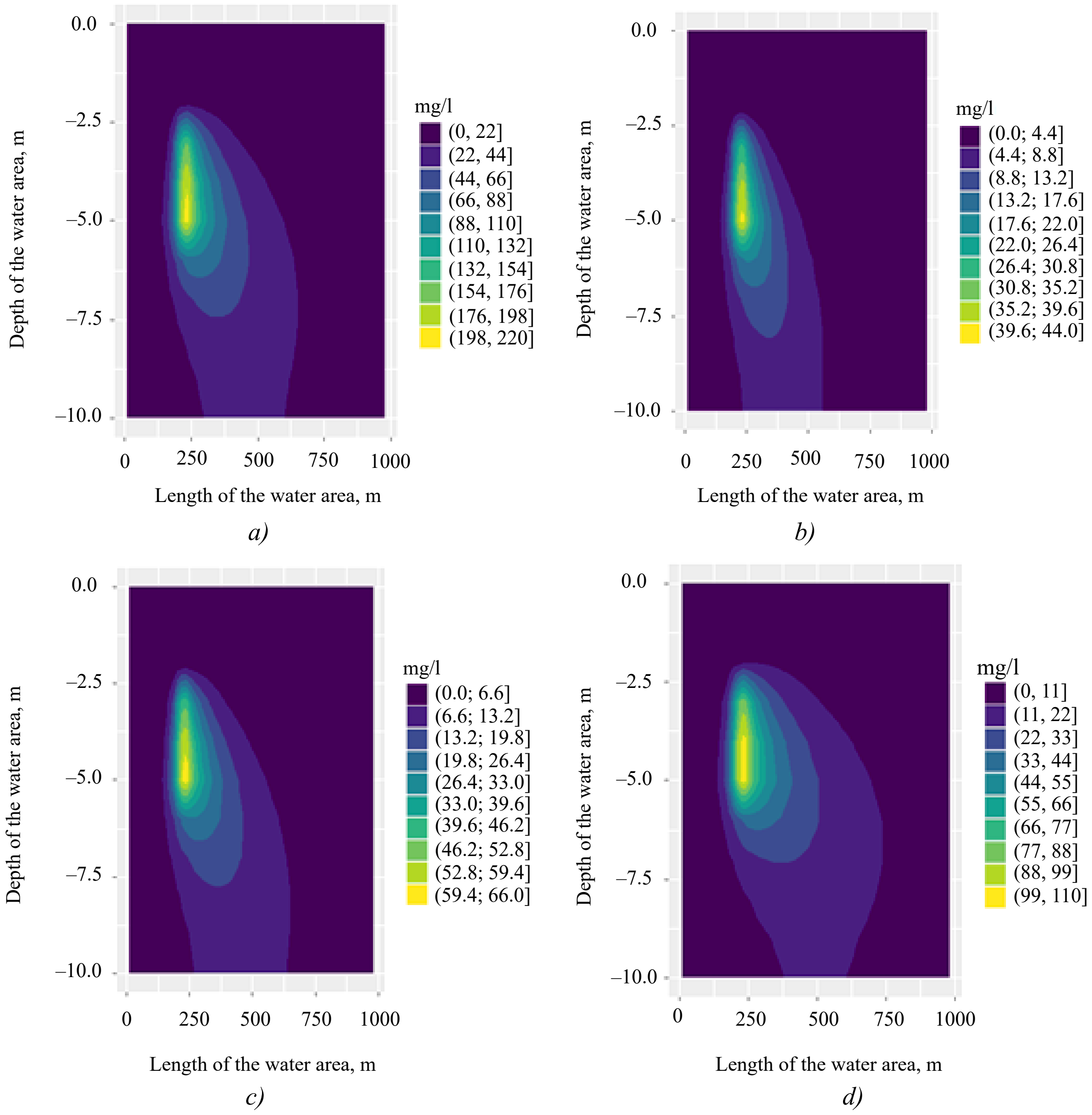

Fig. 1. Concentration of the suspension in the water column 2 hours after unloading

The developed software package allows you to calculate:

– the velocity of the water flow based on the system of equations (2)–(3);

– the process of transport of suspended particles in the water column, taking into account the obtained flow velocity of the water flow;

– the process of settling the suspension on the bottom based on the model (1)–(3).

As an example of the work of the software package, we present the results of numerical modeling of the problem of transport of three-component suspension when modeling the process of dumping soil during dredging.

Parameters of the calculated area: length — 1 km; width — 720 m; depth — 10 m.

The parameters of the calculated grid: the steps along the horizontal and vertical spatial coordinates were 10 and 1 m, respectively; the calculated interval was 2 hours, the time step was 5 seconds. The O_x axis is directed along the calculated area, the O_y axis is along the width of the calculated area, the O_z axis is along the depth of the calculated area (from 0 to –10 m, where the mark 0 corresponds to the water surface, –10 to the bottom of the reservoir).

Input parameters of the model: the average distance from the point of unloading the soil to the bottom of the reservoir in the area of dredging is 5.5 m; the area of unloading the soil along the O_x axis (along the length of the reservoir) is placed in the range from 200 to 250 m; the area of unloading the soil along the O_y axis (along the width of the reservoir) is placed in the range from 300 to 400 m; the flow velocity at depths from 4 to 10 m was 0.075 m/s (currents are directed from left to right); the density of fresh water under normal conditions is 1000 kg/m^3; the density of suspensions is 2700 kg/m^3; the particle shape coefficient for all three suspensions is 0.2222 (spherical shape); the initial viscosity of water is 1.002 MPa /s (at a temperature of 20 °C); the particle diameter of fraction A is 0.05 mm; the deposition rate of fraction A is 2.31 mm/s; the percentage of fraction A is 20 %; the particle diameter of fraction B — 0.04 mm; deposition rate of fraction B — 1.48 mm/s, percentage of fraction B — 30 %; particle diameter of fraction C — 0.03 mm; deposition rate of fraction C — 0.83 mm/s, percentage of fraction C — 50 %.

Fig. 1 shows the results of modeling the process of transport of three-component suspension in the water column. The horizontal axis is directed along the flow, the slice is presented in the middle of the calculated area, where the maximum concentration of suspended particles is observed (in the y = 360 m plane).

Fig. 1 shows that the heavier fraction A is deposited closer to the dredging zone than the lighter fractions B and C. The smaller fractions B and C are evenly distributed along the bottom of the water area.

Discussion and Conclusions. The paper presents a three-dimensional mathematical model of multicomponent suspension transport, supplemented by a three-dimensional model of hydrodynamics of a shallow reservoir. The presented model takes into account the composition of the soil (different diameter of the suspended particles), the flow rate of the water flow, the complex geometry of the coastline and bottom, overburden phenomena, wind currents and friction on the bottom, turbulent exchange, which allows to increase the accuracy of modeling.

The approximation of the proposed multicomponent suspension transport model based on the three-dimensional diffusion-convection equation is performed with the second order of accuracy relative to the steps of the spatial grid, taking into account the boundary conditions of the second and third kind. Approximation of a three-dimensional mathematical model of hydrodynamics is performed on a uniform rectangular computational grid using splitting schemes for physical processes.

For the numerical solution of the obtained discrete models, a software package has been developed that allows simulating the deposition of suspended particles of various diameters on the bottom, and assessing its effect on the bottom relief and changes in the composition of the bottom. The developed software package also allows you to analyze the process of sediment movement in the case of agitation of multicomponent bottom sediments of the reservoir, causing secondary pollution of the reservoir.

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