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Comparison of Hydrodynamic Processes Modelling Results in Shallow Water Bodies Based on 3D Model and 2D Model Averaged by Depth

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Abstract

Introduction. Two-dimensional hydrodynamic models have proven their ability to adequately describe the processes of runoff and transportation in rivers, lakes, estuaries, deltas and seas. Practice shows that even where significant three-dimensional effects are expected, for example, with wind flows, a two-dimensional approach can work effectively. However, in some cases, the two-dimensional model does not accurately reflect the actual flow structures. For example, in shallow waters with complex bathymetry, heterogeneous terrain and dynamics can lead to a non-uniform velocity profile. The aim of the study is to develop a basis for determining in which cases a two-dimensional model averaged in depth is sufficient for modelling hydrodynamic processes in shallow waters like the Azov Sea, and in which cases it is advisable to use a three-dimensional model to obtain accurate results.

Materials and Methods. Local analytical solutions have been obtained for the propagation of the predominant singular progressive wave in a shallow, well-mixed reservoir. Advective terms and Coriolis terms are neglected, the vortex viscosity is assumed to be constant, and the lower friction term is linearized. Special attention is paid to the latter, since the characteristics of the models significantly depend on the method of determining the coefficients of lower friction. The analytical method developed in the study shows that certain combinations of higher flow velocities ($u \approx 1$ m/s) and water depths ($d > 50$ m) can cause significant differences between the results of the depth-averaged model and the model containing vertical information.

Results. The results obtained are verified by numerical simulation of stationary and non-stationary periodic flows in a schematized rectangular basin. The results obtained as a result of three-dimensional modelling are compared with the results of two-dimensional modelling averaged in depth. Both simulations show good compliance with analytical solutions.

Discussion and Conclusions. Analytical solutions were found by linearization of the equations, which obviously has its limitations. A distinction is made between two types of nonlinear effects — nonlinearities caused by higher-order terms in the equations of motion, i. e. terms of advective acceleration and friction, and nonlinear effects caused by geometric nonlinearities, this is due, for example, to different water depths and reservoir widths, which will be important when modelling a real sea.

Keywords: hydrodynamics, shallow water reservoir, wave motion, numerical modelling

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Сопоставление результатов численного моделирования процессов гидродинамики в мелководных водоемах на основе трехмерной модели и двумерной модели, усредненной по глубине

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Аннотация

Введение. Двумерные гидродинамические модели доказали свою способность адекватно описывать процессы стока и транспортировки в реках, озерах, эстуариях, дельтах и морях. Практика показывает, что даже там, где ожидаются значительные трехмерные эффекты, например, при ветровых потоках, двумерный подход может работать эффективно. Однако в некоторых случаях двумерная модель недостаточно точно отражает фактические структуры потока. Например, в мелководных водоемах со сложной батиметрией неоднородный рельеф и динамика могут привести к тому, что профиль скорости будет неоднородным. Целью исследования является разработка основы для определения того, в каких случаях двумерной модели, усредненной по глубине, достаточно для моделирования процессов гидродинамики в мелководных водоемах, подобных Азовскому морю, а в каких случаях для получения точных результатов целесообразно использование трехмерной модели.

Материалы и методы. Локальные аналитические решения получены для распространения преобладающей сингулярной прогрессивной волны в мелководном, хорошо перемешанном водоеме. Адвективными слагаемыми и слагаемыми Кориолиса пренебрегают, вихревая вязкость принимается постоянной, а слагаемое нижнего трения линеаризуется. Последнему уделяется особое внимание, поскольку характеристики моделей существенно зависят от способа определения коэффициентов нижнего трения. Аналитический метод, разработанный в исследовании, показывает, что определенные комбинации более высоких скоростей течения ($u \approx 1$ м/с) и глубин воды ($d > 50$ м) могут вызывать значительные различия между результатами модели, усредненной по глубине, и модели, содержащей информацию по вертикали.

Результаты исследования. Полученные результаты проверяются численным моделированием стационарных и нестационарных периодических течений в схематизированном прямоугольном бассейне. Результаты, полученные в результате трехмерного моделирования, сравниваются с результатами двумерного моделирования, усредненного по глубине. Оба моделирования показывают хорошее соответствие аналитическим решениям.

Обсуждение и заключения. Аналитические решения были найдены путем линеаризации уравнений, что, очевидно, имеет свои ограничения. Отмечается два вида нелинейных эффектов — вызванных членами более высокого порядка в уравнениях движения, т. е. членами адвективного ускорения и трения и вызванных геометрическими нелинейностями, что связано, например, с различной глубиной воды и шириной водоема, что будет важно при моделировании реального моря.

Ключевые слова: гидродинамика, мелководный водоем, волновое движение, численное моделирование

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Introduction. Heterogeneous relief and dynamics in shallow waters with complex bathymetry can lead to the fact that the velocity profile will be heterogeneous, in this case, the two-dimensional model does not accurately reflect the actual flow structures. The main purpose of this study is to develop a theoretical basis for determining in which specific cases a two-dimensional model is sufficient to simulate flow processes in shallow seas such as the Azov Sea. To this end, it has been studied whether the reduction from 3D to 2D has a significant impact on the output data of the hydrodynamics model, such as water levels and depth-averaged flow velocities. It has been studied which simplifications are applied to the flow problem to allow the use of two-dimensional equations averaged over depth, rather than three-dimensional equations for shallow water. It is investigated which parameters are important when comparing a depth-averaged model with a model containing vertical information; in which cases a two-dimensional depth-averaged model is representative. The analytical approach made it possible to identify the effect of depth averaging using analytical solutions for highly simplified situations.

Despite conducting a wide range of studies focused on the problem under consideration, they do not fully reflect the totality of various factors and processes: hydrophysical, hydrodynamic, hydrobiological, meteorological and anthropogenic [1, 2]. The areas of application of the models presented in the study are tides caused by wind (for example, storm surges). River flows and stratified flows determined by density are beyond the scope of this study [3–5].

Let's consider the most important phenomena for shallow seas like the Azov Sea and determine the characteristic scales of length, time and speed. In the Azov Sea, the formation of currents is mainly due to three factors: the wind regime, the flow of flowing rivers and water exchange with the Black Sea. The great variability of currents is a consequence of the instability of the wind regime, the shallow waters of the sea and its relatively small area. In open areas of the sea, under the influence of wind, as a rule, there is a progressive movement of the aquatic environment, covering the entire thickness from the surface to the bottom horizons. The prevailing current velocity in the Azov Sea is 10–20 cm/s, the maximum velocity is 180–100 cm/s [6].

The mode of excitement of the Azov Sea is due to the small area of the sea, shallow depths and significant indentation of the shores. Wave heights of less than 1 m prevail in the described area (their repeatability reaches 75 %). The repeatability of wave heights of 1–2 m is 20–45 %, and wave heights of 2–3 m — no more than 13 %. In the central, deepest part of the sea, wave heights do not exceed 3.5 m and only in very rare cases they reach 4 m. In the stormiest months (December–March), the development of unrest in the described area is limited by the presence of ice [6]. As for overburden phenomena, before which there is often a powerful overclocking, strong waves are characterized by a slight decrease in the height of the waves before the start of the storm and then their rapid increase. Often, the processes of development and attenuation of run-up and storm phenomena in the Azov Sea are synchronous, thus dangerous phenomena arise.

In the Azov Sea, waves are observed with a length of mainly 15–25 m, and only sometimes 80 m. The wave period is usually less than 5 s, extremely rarely — 7–8 s. In the described area, short and very steep waves are noted, which pose a danger to small vessels.

In the shallow Azov Sea, seismic fluctuations occur constantly. Seiches are free fluctuations that occur in moderate-sized basins (harbors, lakes, bays, or even in the sea). These are standing waves with a frequency equal to the resonant frequency of the pool in which they occur. The duration of sessions can vary from a few minutes to several hours. The reason for their appearance in the Azov Sea is not only a change in wind or atmospheric pressure over the sea, but also waves of storm surges from the Black Sea. Since seiches are a resonant phenomenon, it is obvious that the size of the pool in relation to the wavelength is an important factor.

Materials and methods

1. Problem statement. 3D Navier-Stokes equations are used as a basis for solving fluid flow problems [7–8]. Assuming that water is an incompressible liquid, the continuity equation is applied:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0$$

with velocity components u, v, ω in the direction x, y, z respectively. The conservation of momentum is expressed as follows:

$$\begin{aligned} \frac{\partial(pu)}{\partial t} + \frac{\partial(pu^2)}{\partial x} + \frac{\partial(puv)}{\partial y} + \frac{\partial(pu\omega)}{\partial z} &= -\frac{\partial p}{\partial x} + \rho f v + \frac{\partial \tau_{xx}}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{xz}}{\partial z}, \\ \frac{\partial(pv)}{\partial t} + \frac{\partial(puv)}{\partial x} + \frac{\partial(pv^2)}{\partial y} + \frac{\partial(pv\omega)}{\partial z} &= -\frac{\partial p}{\partial y} - \rho f u + \frac{\partial \tau_{yx}}{\partial x} + \frac{\partial \tau_{yy}}{\partial y} + \frac{\partial \tau_{yz}}{\partial z}, \\ \frac{\partial(p\omega)}{\partial t} + \frac{\partial(pu\omega)}{\partial x} + \frac{\partial(pv\omega)}{\partial y} + \frac{\partial(p\omega^2)}{\partial z} &= -\frac{\partial p}{\partial z} - \rho g + \frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{zy}}{\partial y} + \frac{\partial \tau_{zz}}{\partial z}, \end{aligned}$$

where ρ is the water density; p is the pressure; f is the Coriolis parameter; τ_{ij} is the viscous shear stress in the i -direction on the j -plane.

The stresses are defined as:

$$\tau_{ij} = \rho \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right),$$

where ν is the kinematic viscosity.

The abbreviated designation is also used, where $x_i = (x, y, z)$, $u_i = (u, v, \omega)$ for $i = 1, 2, 3$.

To account for turbulence in the Navier-Stokes equations, the variables are decomposed into an average value and variation: $u = \bar{u} + u'$. We substitute the expansion over all variables into the momentum equations and the average results

into the Reynolds-averaged equations of motion, which have the same form as the original Navier-Stokes equations, with additional turbulent stresses called Reynolds stresses: $\tau_{ij} = \rho \overline{u'_i u'_j}$ [9].

The problem of closure is one of the main tasks of turbulence research. A simple turbulence model uses the Boussinesq hypothesis to describe turbulent motions similar to molecular motions, but with vortex viscosity coefficients ν_t^h и ν_t^v for horizontal and vertical directions [10]. All of the above leads to the following equations:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + \omega \frac{\partial u}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial x} + f v + \frac{\partial}{\partial x} \left(\nu_t^h \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left(\nu_t^h \frac{\partial u}{\partial y} \right) + \frac{\partial}{\partial z} \left(\nu_t^v \frac{\partial u}{\partial z} \right), \quad (1)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + \omega \frac{\partial v}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial y} - f u + \frac{\partial}{\partial x} \left(\nu_t^h \frac{\partial v}{\partial x} \right) + \frac{\partial}{\partial y} \left(\nu_t^h \frac{\partial v}{\partial y} \right) + \frac{\partial}{\partial z} \left(\nu_t^v \frac{\partial v}{\partial z} \right), \quad (2)$$

$$\frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} + \omega \frac{\partial \omega}{\partial z} = -\frac{1}{\rho_0} \frac{\partial p}{\partial z} - \frac{\rho}{\rho_0} g + \frac{\partial}{\partial x} \left(\nu_t^h \frac{\partial \omega}{\partial x} \right) + \frac{\partial}{\partial y} \left(\nu_t^h \frac{\partial \omega}{\partial y} \right) + \frac{\partial}{\partial z} \left(\nu_t^v \frac{\partial \omega}{\partial z} \right). \quad (3)$$

For a depth-averaged flow, it is assumed that the shear stress in the layer caused by the turbulent flow is determined by the quadratic law of friction:

$$\tau_b = \frac{\rho_0 g |U| U}{C_1^2} = \rho_0 c_{f1} |U| U,$$

where $|U|$ is the value of the depth-averaged horizontal velocity, and C_1 is the Chezy coefficient for the depth-averaged model [11].

For models with vertical information (2DV or 3D), a quadratic formulation of stresses in the reservoir is used. The shear stress of the layer in 3D can be related to the current directly above the layer:

$$\tau_b = \rho_0 c_{f2} |u_b| u_b,$$

where u_b — the value of the horizontal velocity directly above the layer.

2. Analytical research. Suppose that U characterizes the scales of horizontal velocities u and v , L characterizes the horizontal scales of length x and y , and H characterizes the vertical scale z . Then scaling the continuity equation leads to the expression for the vertical velocity scale W :

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0,$$

$$\frac{U}{L} + \frac{U}{L} + \frac{W}{H} = 0 \Rightarrow W \approx O\left(\frac{UH}{L}\right).$$

According to the data of the Unified State Information System on the Situation in the World Ocean [6], the horizontal component of the velocity is several orders of magnitude greater than the vertical component of the velocity for tides, storm waves and seiche in the Azov Sea. Consequently, the vertical component of the velocity can be neglected, it will not be considered in this study.

The vertical momentum equation reduces to a hydrostatic balance:

$$\frac{\partial p}{\partial z} = -\rho g.$$

Integrating this equation in the vertical direction gives:

$$p(z) = -\rho g(\zeta - z) + p_{atm}, \text{ in case of } \frac{\partial \rho}{\partial z} = 0,$$

where ζ is the height of the free surface (fig. 1); p_{atm} is the atmospheric pressure on a free surface.

Now:

$$\frac{\partial p}{\partial x} = \rho g \frac{\partial \zeta}{\partial x}, \quad \frac{\partial p}{\partial y} = \rho g \frac{\partial \zeta}{\partial y}, \quad \text{если } \frac{\partial \rho}{\partial x} = \frac{\partial \rho}{\partial y} = 0. \quad (4)$$

Substituting this into the horizontal momentum equations (1) and (2), we obtain:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} - \frac{\partial}{\partial z} \left(\nu_t^h \frac{\partial u}{\partial z} \right) = -g \frac{\partial \zeta}{\partial x} + f v, \quad (5)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} - \frac{\partial}{\partial z} \left(\nu_t^h \frac{\partial v}{\partial z} \right) = -g \frac{\partial \zeta}{\partial y} - f u. \quad (6)$$

Along with neglecting the vertical component of velocity and replacing the pressure gradient with the water level gradient, several other terms were ignored in order to move from equations (1) and (2) to (5)–(6). Viscous stresses are not taken into account due to the fact that turbulent stresses are many orders of magnitude higher than viscous stresses, because molecular viscosity is only important within a few millimeters of the boundary. In addition, horizontal turbulent stresses are not taken into account.

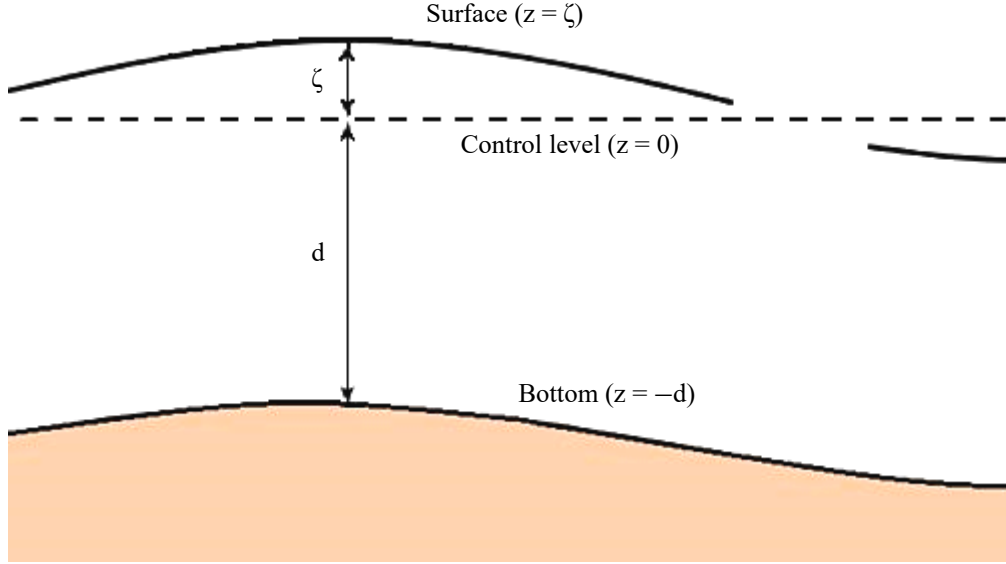


Fig. 1. Determination of surface height ζ and depth of undisturbed water d

Using equations (5)–(6), we determine the relative orders of magnitude of the terms in order to get an idea of the parameters that are most important in a certain situation, and which are insignificant. Let's consider each term separately and perform scaling:

- inertia: $\frac{\partial u}{\partial t} \approx \frac{U}{T}$;
- advection: $u \frac{\partial u}{\partial t} \approx \frac{U^2}{L}$;
- vertical momentum diffusion: $\frac{\partial}{\partial z} \left(\nu \frac{\partial u}{\partial z} \right) \approx \nu \frac{U}{H^2}$;
- pressure gradient: $-g \frac{\partial \zeta}{\partial y} \approx g \frac{\Delta H}{L}$;
- Coriolis force: $f v \approx f U$.

The term responsible for friction is important in the study, since it is defined differently in models. For 3D model, the ratio between the coefficient of friction and the coefficient of inertia can be expressed as:

$$\frac{\text{friction}}{\text{inertia}} \approx \frac{\nu U / H^2}{U / T} \approx \frac{\nu}{\omega H^2},$$

where $\omega \approx \frac{1}{T}$.

In addition, important parameters are viscosity ν , wave frequency ω and vertical length scale (for example, water depth d). A dimensionless combination of these parameters, widely used in the analytical approach, is described in [12], where the dynamic expression for the phase shift stress is compared with the expression in which the bottom stress is proportional to the depth-averaged velocity in the steady state. The result only depending on the parameter $\omega d^2 / \nu$ (fig. 2).

When $\omega d^2 / \nu$ very small, the ratio approaches unity, which means that the shear stress of the layer reacts to the periodic flow as if it were constant at each moment of time: $\omega d^2 / \nu < 0.5$. It can also be written as $d^2 / \nu < 0.5 \cdot T / 2\pi \approx 0.08 \cdot T$, where T is the period of the oscillating flow. The value d^2 / ν can be interpreted as the time required for a viscous flow to change the velocity profile at depth d . Thus, a quasi-stationary shear stress of the layer is expected when the adjustment time is less than 8 % of the period.

For turbulent flow in shallow water, the vortex viscosity estimate is [13]:

$$\nu = \frac{1}{6} \kappa du_* = 0,067 du_*.$$

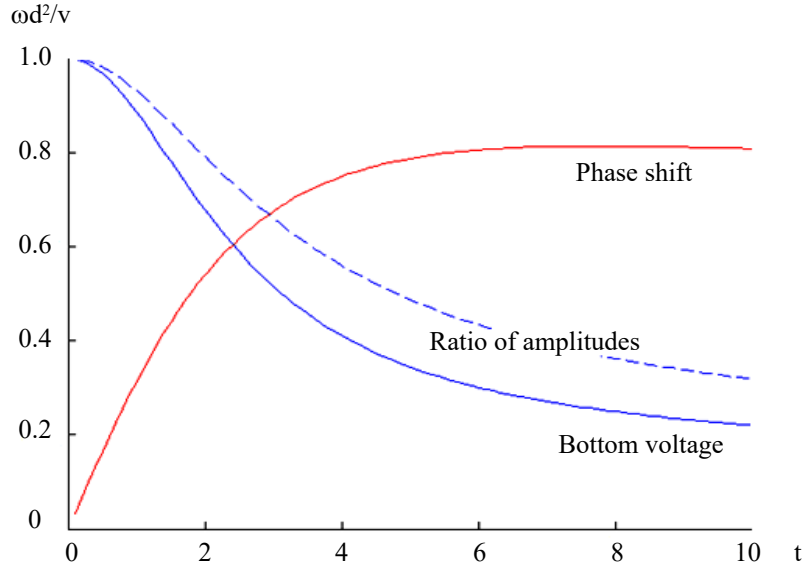


Fig. 2. The ratio of shear stresses of the layer at dynamic and steady flow

For the Azov Sea, with an average water depth $d \approx 7.4$ m and an average amplitude of the current at a depth $\bar{u} \approx 0.5$ m/s, the vortex viscosity estimate is $\nu = 0,00925$ m/s.

Analytical solutions can be obtained only for significantly simplified forms of the equations of motion. Since the advection terms are nonlinear, it is necessary to exclude them from the momentum equation for an analytical approach. When the wave propagation velocity is defined as the wavelength over the wave period $c = L/T$, the ratio between the advection term and the inertia term leads to the Froude number, defined as $F_r = u/c$. $F_r \ll 1$ meaning that the inertia term is much more important than the advection term and therefore the advection term can be neglected. This refers to long waves with small amplitudes relative to the depth of the water. The set of equations without advective terms reduces to:

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu_t \frac{\partial^2 u}{\partial z^2} &= -g \frac{\partial \zeta}{\partial x} + f\nu, \\ \frac{\partial v}{\partial t} - \nu_t \frac{\partial^2 v}{\partial z^2} &= -g \frac{\partial \zeta}{\partial y} - fu, \quad \frac{\partial p}{\partial z} = -\rho g. \end{aligned} \quad (7)$$

We will consider a periodic flow bounded by one horizontal dimension given by equation (7). The main attention will be paid to the effect of changes in the plane (x, z) . Taking into account these assumptions, the basic equation 2DV takes the form:

$$\begin{aligned} \frac{\partial u}{\partial t} - \nu_t \frac{\partial^2 u}{\partial z^2} &= F, \\ F &= -g \frac{\partial \zeta}{\partial x}. \end{aligned} \quad (8)$$

Equation (8) can be integrated over the water depth d to obtain one-dimensional equation of momentum in the x direction:

$$d \frac{\partial U}{\partial t} - \nu_t \left(\frac{\partial \tilde{u}}{\partial z(z=0)} - \frac{\partial \tilde{u}}{\partial z(z=-d)} \right) = Fd,$$

where U is the depth-averaged velocity.

On the sea surface ($z = 0$), the shear stress is zero, since the free surface does not create any friction (in the absence of wind). In the formation ($z = -d$), it is customary to associate the shear stress with the velocity averaged over the depth with the coefficient of friction c_f for the model averaged over the depth. So, when:

$$\nu_t \frac{\partial u}{\partial z} = \tau_b \approx c_{fi} |U| U,$$

the depth-averaged equation takes the form:

$$\frac{\partial U}{\partial t} + \frac{c_{f1}|U|U}{d} = F. \quad (9)$$

Here the friction term is proportional $|U|U$ and, therefore, not linear. Since linear equations are much easier to solve analytically, it is convenient to use the linearization method. Lorenz [14] proposed such a linearization of the shear stress of the layer, which for many decades served as the basis for simple solutions. Suppose that the flow velocity changes sinusoidally in time:

$$U(t) = \hat{U} \cos(\omega t). \quad (10)$$

Only the real part of the expression for U , which is used further, is considered. Figure 3 shows the corresponding $|U|U/\hat{U}^2 = |\cos(\omega t)|\cos(\omega t)$, which represents friction as a function of time, showing deviations from the pure cosine function.

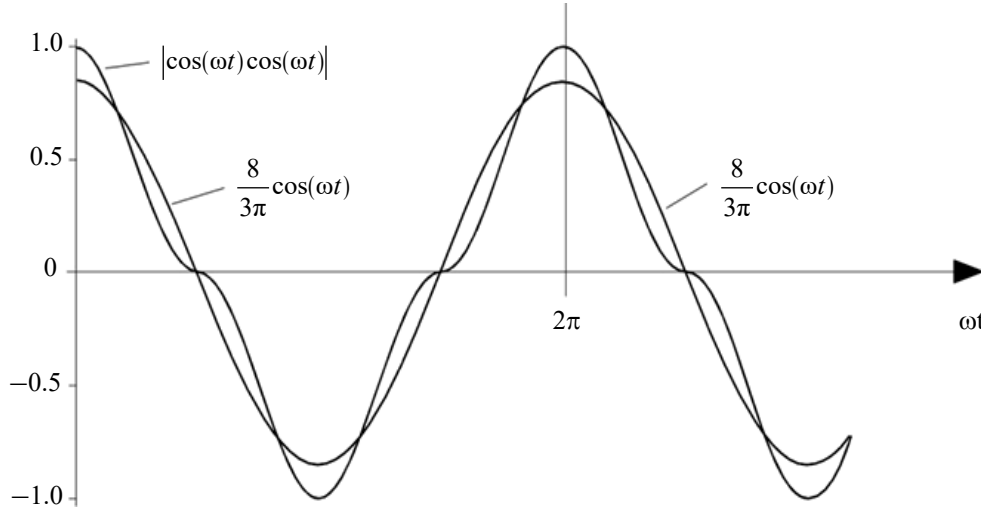


Fig. 3. Linearization of the quadratic friction term [15]

The rationale for linearization lies in the fact that, as is commonly believed, it does not reproduce the exact cosine function as long as the damping effect of friction persists. For this purpose, the energy that is lost per cycle due to friction is set equal for both cases. This approach allows us to obtain a suitable estimate of the linearization constant κ :

$$\kappa_1 = \frac{8}{3\pi} c_{f1} \frac{\hat{U}}{d} \approx c_{f1} \frac{|U|}{d},$$

where the subscript “1” indicates a one-dimensional case. This expression contains the original velocity $\hat{U}$, which is unknown. Although iterative approaches to its definition have been proposed, this linearization constant is often accepted as a calibration parameter.

The shear stress of the linearized layer becomes: $\tau_b = c_{f1}|U|U \approx \kappa_1 d U$.

Substituting this into equation (9), we obtain:

$$\frac{\partial U}{\partial t} + \kappa_1 U = F. \quad (11)$$

For dynamic flow in a one-dimensional situation, this linearized equation is used, while a complex representation of the flow velocity is introduced:

$$U(t) = \hat{U} e^{i\omega t}, \quad (12)$$

where $\hat{U}$ is the complex amplitude; $e^{i\omega t} = \cos(\omega t) + i \sin(\omega t)$, where i is an imaginary unit satisfying the equation $i = \sqrt{-1}$. The factor $e^{i\omega t}$ causes rotation in time with ω as the angular frequency, which is equal to $2\pi/T$.

Note that the velocity defined in (10) is equal to the real part of the expression:

$$U(t) = \text{Re}[\hat{U} e^{i\omega t}] = \hat{U} \cos(\omega t).$$

Substituting the complex periodic solution (12) into equation (11), excluding the time change $e^{i\omega t}$ in each term gives:

$$i\omega\hat{U} + \kappa_1\hat{U} = \hat{F}, \quad (i\omega + \kappa_1)\hat{U} = \hat{F}, \quad \hat{U} = \frac{\hat{F}}{i\omega + \kappa_1}, \quad \hat{U} = \frac{\hat{F}}{i\omega} \cdot \frac{1}{1 + \kappa_1/i\omega}, \quad \hat{U} = \hat{A} \cdot \frac{1}{1 - i\sigma_1},$$

where $\hat{A} = \frac{\hat{F}}{i\omega} = -\frac{g}{i\omega} \frac{\partial \hat{\zeta}}{\partial x}$; $\sigma_1 = \frac{\kappa_1}{\omega} = \frac{8}{3\pi} c_{f_1} \frac{\hat{U}}{\omega d}$.

So, the expression for the depth-averaged velocity in the x direction is:

$$U = \hat{A} \cdot \left(\frac{1}{1 - i\sigma_1} \right) e^{i\omega t}.$$

Thus, the expression for the depth-averaged velocity U is obtained by solving the depth-averaged momentum equation. Next, let's compare this expression with a similar solution for the depth-averaged velocity calculated using a model with vertical information.

The momentum equation $\frac{\partial u}{\partial t} - \nu_t \frac{\partial^2 u}{\partial z^2} = F$ with a periodic solution $u(z, t) = \tilde{u}(z) e^{i\omega t}$ (now $\tilde{u}$ is the complex amplitude as a function of the vertical coordinate) and elimination $e^{i\omega t}$ takes the form:

$$i\omega \tilde{u}(z) - \nu_t \frac{\partial^2 \tilde{u}(z)}{\partial z^2} = \hat{F}$$

and is a differential equation with a homogeneous and partial solution:

$$\tilde{u}(z) = C_1 e^{bz} + C_2 e^{-bz} + \hat{A}, \quad b = \sqrt{\frac{i\omega}{\nu_t}}, \quad \hat{A} = \frac{\hat{F}}{i\omega}.$$

On the sea surface ($z = 0$) shear stress $\tau = 0$ since the free surface does not create friction (in the absence of wind):

$$\begin{aligned} \frac{\partial \tilde{u}}{\partial z} \Big|_{(z=0)} &= 0 \Rightarrow C_1 = C_2 \Rightarrow \\ \Rightarrow \tilde{u}(z) &= C \cosh(bz) + \hat{A}. \end{aligned} \quad (13)$$

There are several variants of the boundary condition in the layer to find an expression for the C integration constant. The partial slip condition assumes velocity in the layer $u \neq 0$. It is assumed that the shear stress in the layer ($z = -d$) is described by the linearized quadratic law of friction:

$$\nu_t \frac{\partial \tilde{u}}{\partial z} \Big|_{(z=-d)} = \tau_b \approx \kappa_2 d \tilde{u},$$

where $\kappa_2 = \frac{8}{3\pi} c_{f_2} \frac{\tilde{u}_b}{d}$.

The subscript "2" indicates a two-dimensional case. Substituting the expression for the flow velocity (13) into equation (14), we obtain:

$$\begin{aligned} \nu_t \cdot (Cb \sinh(-bd)) &= \kappa_2 d \cdot (C \cosh(-bd) + \hat{A}), \\ C \cdot (-\nu_t b \sinh(-bd) - \kappa_2 d \cdot \cosh(bd)) &= \kappa_2 d \hat{A} \Rightarrow \\ \Rightarrow C &= -\hat{A} \cdot \left(\frac{\nu_t b}{\kappa_2 d} \sinh(bd) + \cosh(bd) \right)^{-1}. \end{aligned}$$

The solution for the vertical velocity profile becomes:

$$\tilde{u}(z) = \hat{A} \cdot \left(1 - \frac{\cosh(bz)}{\frac{\nu_t b}{\kappa_2 d} \sinh(bd) + \cosh(bd)} \right), \quad (15)$$

$$\tilde{u}(z) = \hat{A} \cdot \left(1 - \tilde{\gamma} \frac{\cosh(bz)}{\cosh(bd)} \right),$$

where $\tilde{\gamma} = \left(1 + \frac{\nu_t b}{\kappa_2 d} \tanh(bd) \right)^{-1} = \left(1 + \frac{i}{\sigma_2 bd} \tanh(bd) \right)^{-1}$.

So, the velocity profile described by equation (15) is a function of a dimensionless parameter:

$$bd = \sqrt{i\omega d^2/\nu_t}, \quad (16)$$

and dimensionless parameter σ_2 :

$$\sigma_2 = \frac{\kappa_2}{\omega} = \frac{8}{3\pi} c_{f_2} \frac{\hat{u}_b}{\omega d}.$$

The parameter similar to the parameter in equation (16) has already been discussed in the context of the shear stress of the layer $\omega d^2/\nu$.

The depth-averaged velocity is obtained:

$$\begin{aligned} \bar{u} &= \frac{1}{d} \int_{-d}^0 \tilde{u}(z) dz = \frac{1}{d} \int_{-d}^0 \hat{A} \left(1 - \frac{\tilde{\gamma} \cosh(bz)}{\cosh(bd)} \right) dz = \frac{\hat{A}}{d} \left[z - \frac{\tilde{\gamma}}{b} \frac{\cosh(bz)}{\cosh(bd)} \right]_{-d}^0; \\ \bar{u} &= \hat{A} \cdot \left(1 - \frac{\tilde{\gamma}}{bd} \tanh(bd) \right) e^{i\omega t}. \end{aligned}$$

The equation for a one-dimensional steady-state flow model with linearized bottom friction (11) reduces to:

$$\kappa_1 U = F, \quad (17)$$

where $\kappa_1 \approx c_{f_1} |U|/d$ is the linearized coefficient of friction for a depth-averaged flow (for a steady flow, the coefficient $8/3\pi$ is not taken into account); U is the depth-averaged horizontal velocity and $F = -g\partial\zeta/\partial x$ is an abbreviated designation of the water level gradient. The expression for the depth-averaged velocity U follows from equation (17):

$$U = \frac{F}{\kappa_1}. \quad (18)$$

In a two-dimensional steady-state flow model with linearized bottom friction, equation (18) reduces to:

$$-\nu_t \frac{\partial^2 u}{\partial z^2} = F.$$

Since it is assumed that F is independent of z , integration gives:

$$\nu_t \frac{\partial u}{\partial z} = -Fz + C_1.$$

On the sea surface ($z = 0$) the shear stress $\tau_0 = \nu_t \partial u / \partial z = 0$ means that the integration constant $C_1 = 0$ and, therefore:

$$\nu_t \frac{\partial u}{\partial z} = -Fz. \quad (19)$$

Thus, the shear stress is linearly distributed vertically with the maximum shear stress τ_b in the layer ($z = -d$):

$$\tau_b = Fd. \quad (20)$$

Assuming a constant vertical vortex viscosity, the velocity profile is determined by integrating equation (19):

$$u(z) = C_2 - \frac{Fz^2}{2\nu_t}. \quad (21)$$

This is a parabolic velocity profile, where the integrating constant C_2 is equal to the maximum velocity on the sea surface ($z = 0$). To find the constant C_2 , it is necessary to impose a boundary condition on the layer. Various approaches are possible to implement the lower boundary condition.

The combination of the linearized boundary condition $\tau_b = \kappa_2 du_b$, $\kappa_2 \approx c_{f_2} |u_b|/d$ and equation (20) will result in an expression for the velocity in the layer. Substituting this into equation (21), we obtain an expression for C_2 . Then the speed profile becomes:

$$u(z) = \frac{F}{2\nu_t} (d^2 - z^2) + \frac{F}{\kappa_2}.$$

Integration by depth gives:

$$\bar{u} = \frac{1}{d} \int_{-d}^0 u(z) dz = \frac{1}{d} \int_{-d}^0 \left(\frac{F}{2\nu_t} (d^2 - z^2) + \frac{F}{\kappa_2} \right) dz = \frac{Fd^2}{3\nu_t} + \frac{Fd^2}{\kappa_2}. \quad (22)$$

Thus, in the stationary case, there are also two solutions, (18) and (22), with two different bottom friction coefficients, where $\kappa_1 \neq \kappa_2$ since the shear stress of the layer is determined differently in both cases. Assume that the depth-averaged velocities for both models are equal $\bar{u} = U$. This leads to the following relation between κ_1 and κ_2 :

$$\frac{1}{\kappa_1} = \frac{1}{\kappa_2} + \frac{d^2}{3\nu_t}.$$

Applying $\sigma = \kappa / \omega$:

$$\frac{1}{\sigma_1} = \frac{1}{\sigma_2} + \frac{\omega d^2}{3\nu_t}. \quad (23)$$

In the relation between σ_1 and σ_2 there is a dimensionless parameter $\omega d^2/\nu$ now choosing a certain range for σ_1 will result in obtaining σ_2 as dimensionless parameter function for each σ_1 . σ -ratio (23) helps comparing the amplitudes and phases of depth-averaged velocities calculated using a one-dimensional and two-dimensional model, respectively.

Figures 4 and 5 show the ratio of the velocity amplitude and phase for several values σ_1 . The relations of amplitudes and phases for the corresponding physical phenomena are indicated by dots. Thus, the application of the σ -relation is limited to situations $\omega d^2/\nu_t > \sigma_1$.

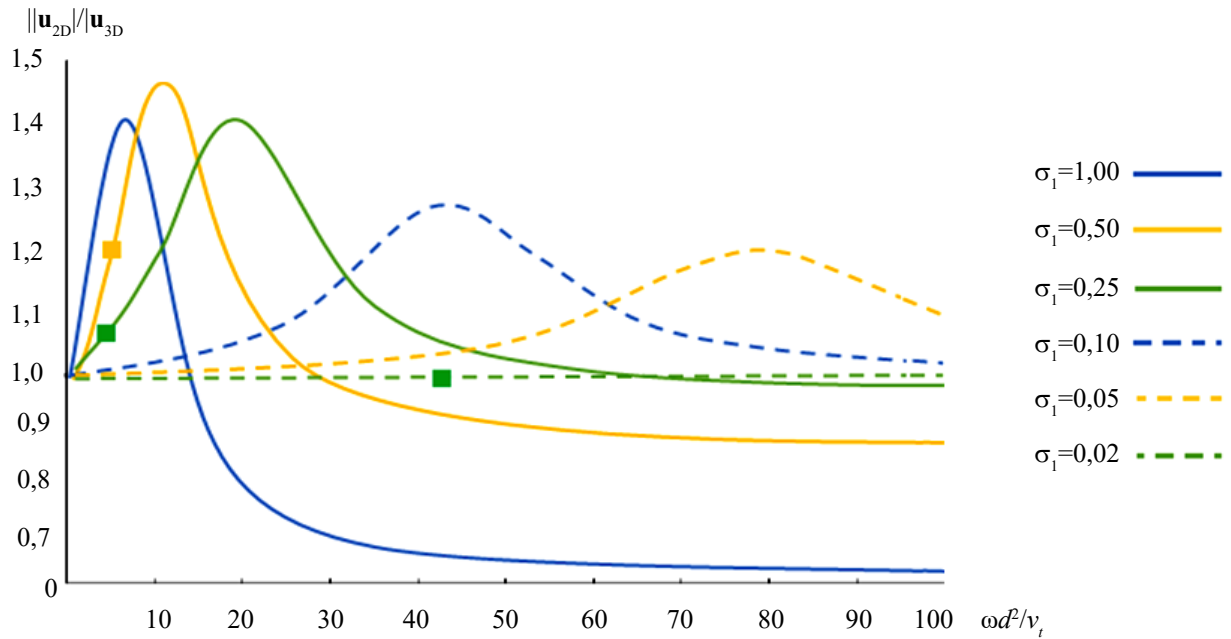


Fig. 4. The ratio of amplitudes between U and $\bar{u}$ as a function $\omega d^2/\nu_t$ and σ_1

The ratio of velocity amplitudes is greater than 1, provided that $\omega d^2/\nu_t < 65$, this means that the velocity calculated using a depth-averaged model will be greater than the amplitude of the velocity that contains information about the vertical than vice versa. In addition, it should be noted that for seas like the Azov Sea (green dot on the line $\sigma_1 = 0.25$) the ratio of the velocity amplitude to 1.06 and the phase ratio to 1.01. An amplitude deviation of 6 % is expected for parameter values that are likely to occur in practice. At the same value $\omega d^2/\nu_t$ but a higher value σ_1 the differences increase (yellow dot on the line $\sigma_1 = 0.50$). The only reason for the increase, when ω , d and ν_t remain constant would be an increase in velocity $\hat{U}$. So, the initial initial velocity was estimated as $\hat{U} = 0.5$ m/s, for $\hat{U} = 1$ m/s the yellow dot indicates the ratio of the velocity amplitudes 1.23 and the phase ratio 0.94.

At lower values of the ratio of velocity amplitudes is almost always greater than 1, regardless of the value $\omega d^2/\nu_t$. In the case of inertia dominance (for example, for a seiche) one should not expect any difference between calculations with vertical information and without it. This is indicated by a green dot on the line $\sigma_1 = 0.02$.

The result of the amplitude-phase ratio seems to be significantly sensitive to the estimation of velocity amplitudes $\hat{U}$, as well as to the bottom friction coefficient c_f . It can be concluded that the regions in which significant differences (more than 20 %) should be expected are difficult to quantify in a general sense, in addition, there are certain combinations of flow velocity and depth water, which can lead to a significant difference in results.

In a one-dimensional steady-state flow model with quadratic bottom friction, the momentum equation reduces to:

$$\frac{c_{f_1}|U|U}{d} = F,$$

where c_{f_1} is the coefficient of friction for the flow averaged over depth.

Since U can be considered positive at steady flow, the expression for the depth-averaged velocity becomes:

$$U = \sqrt{\frac{Fd}{c_{f_1}}}.$$

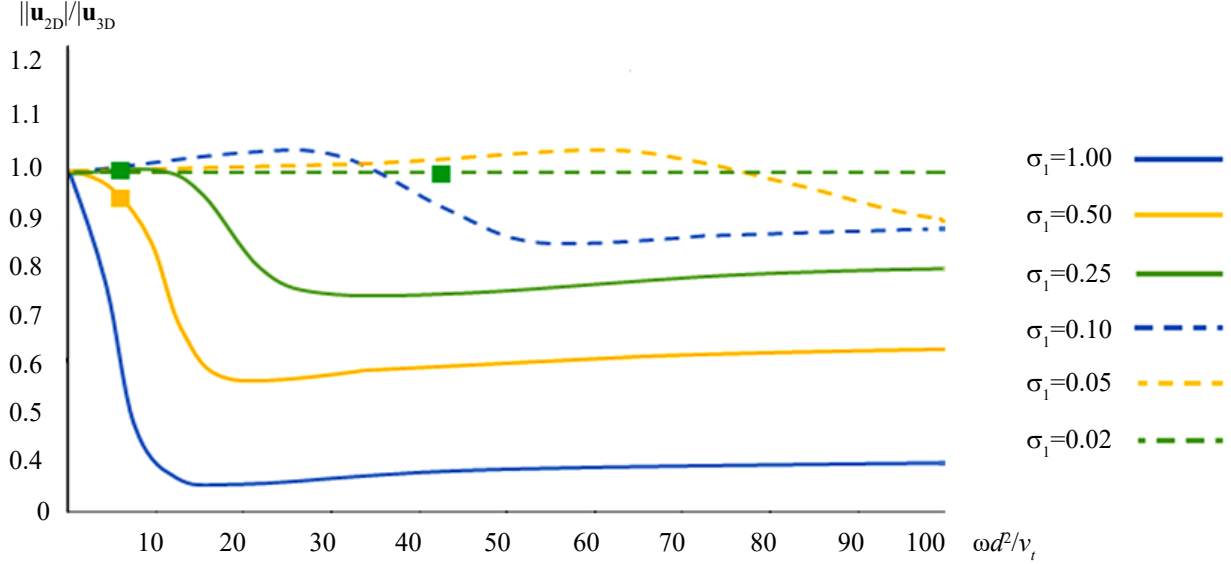


Fig. 5. The phase ratio between U and $\bar{u}$ as a function $\omega d^2/v_t$ and σ_1

The starting point for a steady flow with quadratic bottom friction for a two-dimensional model is equation (21). The integration constant is again found by superimposing a boundary condition on the layer. The combination of the quadratic boundary condition ($\tau_b = c_{f_2}|u_b|u_b$) and equation (20) will result in an expression for the velocity in the layer. Substituting this into equation (21), we obtain an expression for C_2 , thus, the velocity profile becomes:

$$u(z) = \frac{F}{2v_t}(d^2 - z^2) + \sqrt{\frac{Fd}{c_{f_2}}},$$

where c_{f_2} is the coefficient of friction for the model containing information in the vertical direction. Integration by depth gives:

$$\bar{u} = \frac{1}{d} \int_{-d}^0 u(z) dz = \frac{1}{d} \int_{-d}^0 \left(\frac{F}{2v_t}(d^2 - z^2) + \sqrt{\frac{Fd}{c_{f_2}}} \right) dz = \frac{Fd^2}{3v_t} + \sqrt{\frac{Fd}{c_{f_2}}}.$$

Similarly, to the linearized case, under the assumption $\bar{u} = U$ follows the relation between c_{f_1} and c_{f_2} :

$$\frac{1}{\sqrt{c_{f_1}}} = \frac{1}{\sqrt{c_{f_2}}} + \frac{\sqrt{Fd^3}}{3v_t},$$

also $c_{f_1} \neq c_{f_2}$ and c_{f_2} , since the shear stress of the layer is determined differently. The elevation function of the level can be included in this analysis using the continuity equation:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial \omega}{\partial z} = 0.$$

The depth-averaged version of the continuity equation is:

$$\frac{\partial \zeta}{\partial t} + d \frac{\partial U}{\partial x} = 0, \quad (24)$$

where ζ is the height above sea level on a free surface; d is the depth of the water; U is the depth-averaged velocity. The height of the surface can be found by substituting an expression for the depth-averaged velocity calculated using two models.

The previously obtained one-dimensional depth averaged velocity U is equal to:

$$U = A \left(\frac{1}{1 - i\sigma_1} \right) e^{i\omega t},$$

where $A = F / i\omega$ и $F = -g\partial\zeta / \partial x$.

Substituting this into the depth-averaged continuity equation (24), we obtain:

$$i\omega\tilde{\zeta} - \frac{dg}{i\omega} \frac{\partial^2 \tilde{\zeta}}{\partial x^2} \left(\frac{1}{1 - i\sigma_1} \right) e^{i\omega t} = 0.$$

Substituting $\zeta(x, t) = \tilde{\zeta}(x, t) e^{i\omega t}$, we obtain: $i\omega\tilde{\zeta} - \frac{dg}{i\omega} \frac{\partial^2 \tilde{\zeta}}{\partial x^2} \left(\frac{1}{1 - i\sigma_1} \right) e^{i\omega t} = 0$.

This differential equation has the following homogeneous solution:

$$\tilde{\zeta}(x) = C_+ e^{-p_1 x} + C_- e^{p_1 x},$$

where $\pm p_1 = \pm ik_0 \sqrt{1 - i\sigma_1}$, $k_0 = \omega / c_0$ is the wave number without friction; $c_0 = \sqrt{gd}$ is the speed of the wave without friction.

The general periodic solution given by equation (25) contains two exponentially decaying waves propagating in the opposite direction. Here $\pm ik_0$ represents propagation in the absence of friction, $\sqrt{1 - i\sigma_1}$ represents the effect of friction. Rewriting p as $p_1 = \mu + ik$:

$$\text{Im}(p) = k_1 = \frac{k_0}{\sqrt{1 - \tan^2 \delta}}, \quad \text{Re}(p) = \mu_1 = k_1 \tan \delta,$$

where $\tan^2 \delta \equiv \sigma_1 = \frac{k_1}{\omega}$ or $\tan 2\delta \equiv \sigma_1 = \frac{k_1}{\omega}$.

Here, instead of working with σ (the ratio of friction to inertia), the friction angle δ is determined, since this seems more convenient. The general periodic solution of equation (25) can be written as:

$$\tilde{\zeta}(x) = C_+ e^{-p_1 x} + C_- e^{p_1 x}, \quad \tilde{\zeta}(x) = C_+ e^{-\mu_1 x} e^{-i k_1 x} + C_- 2e^{\mu_1 x} e^{-i k_1 x}, \quad \tilde{\zeta}(x) = \tilde{\zeta}_+ + \tilde{\zeta}_-. \quad (26)$$

This is an abbreviated designation for two waves propagating in the opposite direction.

Similarly, u substituted into the continuity equation, which leads to the following complex root:

$$\pm p_2 = \pm ik_0 \left(1 - \frac{\tilde{\gamma}}{bd} \tanh(bd) \right)^{-1/2},$$

where $\tilde{\gamma} = \left(1 + \frac{i}{\sigma_2 bd} \tanh(bd) \right)^{-1}$, $\sigma_2 = \frac{\kappa_2}{\omega}$, $bd = \sqrt{\frac{i\omega}{\nu_t}} d$.

The following section illustrates some of the previous developments for a singular translational wave and waves in a pool closed at one end, both for the two-dimensional and three-dimensional case.

Results. To interpret the solution for a singular progressive wave, we will use:

$$\zeta(x, t) = \text{Re} \{ \tilde{\zeta} e^{i\omega t} \}.$$

A singular traveling wave is considered, so we will use from equation (26):

$$\zeta_+(x, t) = \text{Re} \{ \tilde{\zeta}_+(x) e^{i\omega t} \} = \text{Re} \{ C_+ e^{-p_1 x} e^{i\omega t} \}.$$

Substituting $p_2 = \mu + ik$ and taking C_1 modulo and argument, we get:

$$\zeta_+(x, t) = \text{Re} \{ |C_+| e^{-\mu x} e^{i(\omega t - kx + \arg C_+)} \} \quad \text{or} \quad \zeta_+(x, t) = \tilde{\zeta}_+(x) e^{i(\omega t - kx + \arg C_+)},$$

where $\tilde{\zeta}_+(x) = |C_+| e^{-\mu x}$.

This solution shows the surface height of a singular translational wave with k (the imaginary part of p) as the wave number (phase change per unit length). Since the phase varies with x and t through $\omega t - kx$, this refers to a progressive wave in the positive direction x , with a phase velocity $c = \omega/k$ an argument C_+ is the initial phase (phase when $\omega t = 0$) ζ_+ for $x = 0$. The amplitude $\zeta_+(x, t)$ for $x = 0$ is given through $|C_+|$ decreases exponentially in the positive direction along the x axis with a decay rate of μ .

The results of numerical experiments are presented in Table 1. Case 1 illustrates a situation in which bottom friction does not significantly contribute to the solution ($\sigma_1 = 0.05$; $d = 60$ m); for case 2, friction is essential ($\sigma_1 = 0.50$; $d = 20$ m).

Table 1

Calculated parameters of a wave with constant viscosity $\nu_t = 0.05$ m²/s

Parameter	Formula	Case 1	Case 2
σ_1		0.05	0.50
σ_2		0.06	0.62
d		60 [m]	20 [m]
c_0	$\sqrt{gd}$	24.3 [m/c]	14.0 [m/c]
k_0	ω / c	$5.80 \cdot 10^{-6}$	$1.00 \cdot 10^{-5}$
k_1	$\text{Im}(p_1)$	$5.80 \cdot 10^{-6}$	$1.03 \cdot 10^{-5}$
k_2	$\text{Im}(p_2)$	$5.81 \cdot 10^{-6}$	$8.79 \cdot 10^{-6}$
μ_1	$\text{Re}(p_1)$	$1.45 \cdot 10^{-7}$	$2.44 \cdot 10^{-6}$
μ_2	$\text{Re}(p_2)$	$1.50 \cdot 10^{-7}$	$1.56 \cdot 10^{-6}$

The amplitude of the surface rise decreases with a coefficient $e^{-i\mu\Delta x} \cdot \exp(-\mu\Delta x)$. At a distance of $\Delta x = 10$ km one-dimensional reduction coefficient for case 1 is equal $e^{(-0.015)} \approx 0.99$, two-dimensional one is also equal $e^{(-0.015)} \approx 0.99$. For case 2, the one-dimensional reduction coefficient is $\exp(-0.24) \approx 0.78$, 2D — $\exp(-0.16) \approx 0.86$. It can be concluded that the differences between one-dimensional and two-dimensional results are not significant when bottom friction does not significantly contribute to the solution.

Discussion and Conclusion. Analytical solutions for a depth-averaged model and a model that contains vertical information are:

$$U = \tilde{A} \cdot \left(\frac{1}{1 - i\sigma_1} \right) e^{i\omega t}, \quad \bar{u} = \tilde{A} \cdot \left(1 - \frac{\tilde{\gamma}}{bd} \tanh(bd) \right) e^{i\omega t},$$

where $\tilde{\gamma}$ is the function of σ_2 and bd . Thus, the depth-averaged velocities in both models look very similar and can be described by a function of a dimensionless σ_1 -parameter or σ_2 -parameter and a dimensionless parameter bd , respectively, where:

$$\sigma_1 = \frac{8}{3\pi} c_{f1} \frac{\bar{U}}{\omega d}, \quad \sigma_2 = \frac{8}{3\pi} c_{f2} \frac{\tilde{u}_b}{\omega d}, \quad bd = \sqrt{\frac{i\omega d^2}{\nu_t}}.$$

In order to link the above solutions, it was decided to assume that the depth-averaged velocities in both cases are equal for a steady flow.

Concentrating on the propagation of one predominant singular progressive wave, the analytical approach shows that certain conditions can cause significant differences between the depth-averaged velocities calculated using two-dimensional and three-dimensional models. However, a thorough study has led to the conclusion that it is quite difficult to find such conditions in practice. Thus, in combination with the uncertainties associated with the relationship between the two models, the greatest differences should be expected in places with greater water depth ($d > 60$ m) and high speeds ($u \gg 1$ m/s). It should also be noted that the ratio of amplitudes is almost always greater than 1, and the ratio of phases < 1 for seas.

Analytical solutions were found by linearization of the equations, which obviously has its limitations. A distinction is made between two types of nonlinear effects:

1. Non-linearities caused by higher-order terms in the equations of motion, i.e. the terms of advective acceleration and friction. The linearization of the kU friction term is based on optimal reproduction of the prevailing singular progressive wave. Although such linearization is effective for the purposes of this study, it distorts the propagation and generation of other components of the motion of the aquatic environment.

2. Nonlinear effects caused by geometric nonlinearities that result from the dependence of the cross-section on the height of the surface ζ . This is due, for example, to the different depth of water and the width of the reservoir, which will be important when modelling a real sea.

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