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Original article

Stationary and Non-Stationary Periodic Flows Mathematical Modelling using Various Vortex Viscosity Models

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Abstract
Introduction. Mathematical modelling of currents is an urgent research topic in the field of hydrodynamics and oceanography. Despite ongoing research in the field of developing accurate and efficient numerical methods for solving Navier-Stokes equations that take into account vortex viscosity, the problems of accurate prediction and control of turbulence remain unresolved. The influence of nonlinear effects in vortex viscosity models on the accuracy of forecasts and their applicability to various flow conditions also remains relevant. The aim of the study is to study the influence of linearized and quadratic bottom friction and two turbulence models on the numerical solution of stationary and non-stationary periodic flows. Special emphasis is placed on comparing numerical results with analytical solutions within the framework of using various models of bottom friction.
Materials and Methods. The computational models used in this study are based on a simplified two-dimensional wave model and full three-dimensional Navier-Stokes equations. The classical model of shallow water motion and the 2D model without taking into account dynamic changes in the geometry of the reservoir surface are derived from a system of equations for a spatially inhomogeneous three-dimensional mathematical model of wave hydrodynamics of a shallow reservoir. Analytical solutions were found by linearization of the equations, which obviously has its limitations. A distinction is made between two types of nonlinear effects — nonlinearities caused by higher-order terms in the equations of motion, i. e. terms of advective acceleration and friction, and nonlinear effects caused by geometric nonlinearities, this is due, for example, to different water depths and reservoir widths, which will be important when modelling a real sea.
Results. The results of modelling stationary and non-stationary periodic flows in a schematized rectangular basin using linearized bottom friction are presented. The influence of linearization on the numerical solution is investigated in comparison with analytical profiles using models calculating bottom friction in a quadratic formulation. In combination with quadratic bottom friction, two turbulence models are studied: the constant vortex viscosity and the Prandtl mixing length model. The results obtained as a result of three-dimensional modeling are compared with the results of two-dimensional modelling and analytical solutions averaged in depth.
Discussion and Conclusion. New approaches to modelling and studying flows with variable vortex viscosity are proposed, including analysis of the influence of linearization and the use of various turbulence models. For the linearized and quadratic formulations of bottom friction, it is proved that the numerical results for the case of stationary flow show great similarity with analytical solutions, since the surface height is much less than the water depth and advection can be neglected. The numerical results for the unsteady flow also show a good agreement with the theory. Unlike analytical solutions, numerical modeling has minor deviations in the long run. The study of flows, within the framework of using various turbulence models, will make it possible to take into account the influence of nonlinear effects in vortex viscosity models on the accuracy of forecasts and their applicability to various flow conditions. The results obtained make it possible to better understand and describe the physical processes occurring in shallow waters. This opens up new possibilities for applying mathematical modeling to predict and analyze the impact of human activities on the marine environment and to solve other problems in the field of oceanology and geophysics.

Keywords: hydrodynamics, shallow water reservoir, wave motion, numerical modelling

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Научная статья

Математическое моделирование стационарных и нестационарных периодических течений с использованием различных моделей вихревой вязкости

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Аннотация

Введение. Математическое моделирование течений является актуальной исследовательской темой в области гидродинамики и океанографии. Несмотря на непрекращающиеся исследования в области разработки точных и эффективных численных методов для решения уравнений Навье-Стокса, учитывающих вихревую вязкость, задачи точного предсказания и контроля турбулентности остаются нерешенными. Также актуальными остаются вопросы влияния нелинейных эффектов в моделях вихревой вязкости на точность прогнозов и их применимость к различным условиям течения. Целью исследования является изучение влияния линеаризованного и квадратичного донного трения и двух моделей турбулентности на численное решение стационарных и нестационарных периодических течений. Особый акцент сделан на сравнении численных результатов с аналитическими решениями в рамках использования различных моделей донного трения.

Материалы и методы. Вычислительные модели, применяемые в этом исследовании, основаны на упрощенной двумерной волновой модели и полных трехмерных уравнениях Навье-Стокса. Классическая модель движения мелкой воды и 2D-модель без учета динамического изменения геометрии поверхности водоема получены из системы уравнений для пространственно-неоднородной трехмерной математической модели волновой гидродинамики мелководного водоема. Аналитические решения были найдены путем линеаризации уравнений, что, очевидно, имеет свои ограничения. Проводится различие между нелинейностями, вызванными членами более высокого порядка в уравнениях движения (т. е. членами адвективного ускорения и трения), и геометрическими нелинейностями, связанными, например, с различной глубиной воды и шириной водоема, что будет важно при моделировании реального моря.

Результаты исследования. Представлены результаты моделирования стационарных и нестационарных периодических течений в схематизированном прямоугольном бассейне с использованием линеаризованного донного трения. Исследовано влияние линеаризации на численное решение в сравнении с аналитическими профилями, использующими модели, рассчитывающие донное трение в квадратичной формулировке. В сочетании с квадратичным трением о дно изучаются две модели турбулентности: постоянная вихревая вязкость и модель длины перемешивания Прандтля. Результаты, полученные в результате трехмерного моделирования, сравниваются с результатами двумерного моделирования и аналитическими решениями, усредненными по глубине.

Обсуждение и заключение. Предложены новые подходы к моделированию и исследованию течений с переменной вихревой вязкостью, включая анализ влияния линеаризации и использование различных моделей турбулентности. Для линеаризованной и квадратичной формулировок донного трения доказано, что численные результаты для случая стационарного течения демонстрируют большое сходство с аналитическими решениями, поскольку высота поверхности намного меньше глубины воды и адвекцией можно пренебречь. Численные результаты для нестационарного течения также показывают хорошее соответствие теории. В отличие от аналитических решений численное моделирование имеет незначительные отклонения в долгосрочной перспективе. Исследование течений, в рамках использования различных моделей турбулентности, позволит осуществить учет влияния нелинейных эффектов в моделях вихревой вязкости на точность прогнозов и их применимость к различным условиям течения. Полученные результаты позволяют лучше понять и описать физические процессы, происходящие в мелководных водоемах. Это открывает новые возможности применения математического моделирования для прогнозирования и анализа воздействия человеческой деятельности на морскую среду и для решения других задач в области океанологии и геофизики.

Ключевые слова: гидродинамика, мелководный водоем, волновое движение, численное моделирование

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Introduction. Mathematical modelling of flows is an important and relevant tool for scientific and engineering research, allowing to identify possible risks, optimize processes and investigate complex physical phenomena that are difficult or impossible to study experimentally. The application of the mathematical modeling method makes it possible to study the main characteristics of currents: velocity, pressure, concentration, temperature, which cannot be measured directly. For example, modeling can help predict the spread of pollution in water systems or determine the optimal flood control strategy.

Many scientists are engaged in flow research using various models of vortex viscosity. The analysis of studies [1−10] related to the development of numerical methods aimed at solving the Navier-Stokes equations for complex periodic flows in the field of turbulence and fluid dynamics suggests that modeling stationary and nonstationary periodic flows remains an important scientific and applied problem.

Despite the successes in this direction — the development of more accurate and efficient numerical methods for solving Navier-Stokes equations that take into account vortex viscosity (these methods allow for more accurate modeling of complex flows, such as the flow around bodies with a high degree of vortex activity), there are also unsolved problems. This includes, for example, accurate prediction and control of turbulence. The influence of nonlinear effects in vortex viscosity models on the accuracy of forecasts and their applicability to various flow conditions also remains relevant. Such models make it possible to obtain a more accurate and realistic description of the behavior of the fluid flow. This is especially important when studying turbulent flows, where vortex viscosity is one of the key factors influencing the nature of fluid movement. Modelling of such flows makes it possible to refine the parameters of vortices, determine their effect on other physical processes and develop methods for controlling or controlling the flow.

The use of various vortex viscosity models makes it possible to take into account flow features such as flow geometry, the presence of obstacles, changes in density or viscosity. Each vortex viscosity model has its limitations and its choice depends on specific factors and modeling goals. Comparing the results obtained using different models allows them to be refined and verified, as well as to draw more accurate conclusions about the behavior of the flow.

Materials and Methods. The computational models used in this study are based on a simplified two-dimensional wave model and full three-dimensional Navier-Stokes equations.

A spatially inhomogeneous three-dimensional mathematical model of wave hydrodynamics of a shallow reservoir includes [1]:

– Navier-Stokes equations of motion:

$$u'_t + uu'_x + vu'_y + wu'_z = -\frac{1}{\rho}P'_x + (\mu u'_x)'_x + (\mu u'_y)'_y + (\nu u'_z)'_z,$$

$$v'_t + uv'_x + vv'_y + wv'_z = -\frac{1}{\rho}P'_y + (\mu v'_x)'_x + (\mu v'_y)'_y + (\nu v'_z)'_z, \quad (1)$$

$$w'_t + uw'_x + vw'_y + ww'_z = -\frac{1}{\rho}P'_z + (\mu w'_x)'_x + (\mu w'_y)'_y + (\nu w'_z)'_z + g;$$

– continuity equation:

$$\rho'_t + (\rho u)'_x + (\rho v)'_y + (\rho w)'_z = 0, \quad (2)$$

where $\mathbf{V} = \{u, v, w\}$ is the water flow velocity vector; ρ is the density of the aquatic environment; P is the hydrodynamic pressure; g is the acceleration of gravity; μ, ν are the coefficients of turbulent exchange in horizontal and vertical directions; $\mathbf{n}$ is the vector of the normal to the surface describing the boundary of the computational domain.

2D mathematical model of the motion of the aquatic environment is based on 3D model and includes:

– Navier-Stokes equations:

$$u'_t + uu'_x + vu'_y + wu'_z = -\frac{1}{\rho}P'_x + \left(\mu u'_x\right)'_x + \left(\mu u'_y\right)'_y + \left(\eta u'_z\right)'_z,$$

$$v'_t + uv'_x + vv'_y + wv'_z = -\frac{1}{\rho}P'_y + \left(\mu v'_x\right)'_x + \left(\mu v'_y\right)'_y + \left(\eta v'_z\right)'_z;$$

– continuity equation (for incompressible fluid): $u'_x + v'_y + w'_z = 0$;

– equation of hydrostatics: $P = \rho g\,(z + \xi)$.

In the hydrostatic case, the continuity equation has the form [11, 12]:

$$\theta'_t + (Hu)'_x + (Hv)'_y = 0,$$

where $\theta = \min(\chi, \xi)$; $H = h + \theta$, h is the depth of the reservoir.

From the developed system of equations, it is possible to obtain a classical model of shallow water movement and a 2D model without taking into account the dynamic change in the geometry of the reservoir surface.

Analytical solutions for a depth-averaged model and a model that contains vertical information are:

$$U = \widetilde{A} \cdot \left(\frac{1}{1 - i\sigma_1}\right) e^{i\omega t},$$

$$\bar{u} = \widetilde{A} \cdot \left(1 - \frac{\widetilde{\gamma}}{bd}\tanh(bd)\right) e^{i\omega t},$$

where $\widetilde{\gamma}$ is a function only of σ_2 and bd.

Thus, the depth-averaged velocities in both models look very similar and can be described by a function of dimensionless parameters σ_1, σ_2 and bd respectively, where:

$$\sigma_1 = \frac{8}{3\pi} c_{f_1} \frac{\widehat{U}}{\omega d}, \quad \sigma_2 = \frac{8}{3\pi} c_{f_2} \frac{\widetilde{u}_b}{\omega d}, \quad bd = \sqrt{\frac{i\omega d^2}{\nu_t}}.$$

Analytical solutions were found by linearization of equations, which has its limitations. A distinction is made between two types of nonlinear effects:

1. Non-linearities caused by higher-order terms in the equations of motion, i.e. the terms of advective acceleration and friction. *kU* friction linearization is based on optimal reproduction of the prevailing singular progressive wave. Although such linearization is effective for the purposes of this study, it distorts the propagation and generation of other components of the motion of the aquatic environment.

2. Nonlinear effects caused by geometric nonlinearities that result from the dependence of the cross section on the height of the surface. This is due, for example, to the different water depth and width of the reservoir, which will be important when modelling a real sea.

Turbulence modelling. Turbulent viscosity expresses momentum transfer in a turbulent flow. Several models of turbulent viscosity are available:

− constant vortex viscosity model;

− Prandtl length mixing model;

− k–ε model;

− Large eddy simulation (LES) [4, 7–8].

The constant vortex viscosity model is a simple model describing vortex viscosity as the product of velocity and length scale:

$$\nu_e = \frac{1}{6}\kappa d u_*.$$

The Prandtl mixing length model uses the mixing length hypothesis, in which the velocity characterizing turbulent fluctuations is proportional to the velocity difference in the average flow at a distance l_m, at which mixing or momentum transfer occurs, and is determined as $l_m \cdot \frac{\partial \bar{u}}{\partial x} \cdot$ when reused l_m as a control length scale, the vortex viscosity can be written as the product of this scale squared by the local velocity gradient [13–15].

The model k–ε relates the turbulence viscosity to the kinetic energy of turbulence k and the velocity of turbulence dispersion. The evolution of k and ϵ in time is described by the transfer equations.

When working with large coherent turbulent structures, the method of Large eddy simulation (LES) should be used. In LES models, large turbulence scales are directly resolved on the computational grid, while smaller scales are accounted for using the closure formulation.

The simulation is performed using the following boundary conditions:

− losed boundaries on the bottom, embankment or wall (with or without wall friction);

− free surface boundary;

− constant water level at open borders;

− harmoniously changing water level.

The coefficient of friction in the depth-averaged model (c_{f1}), differed from the coefficient of friction in the vertical information model (c_{f2}), while a constant vertical vortex viscosity was used. In practice, numerical modelling usually uses a viscosity that varies vertically in accordance with the turbulent mixing model along the length. Using this definition of vortex viscosity and integrating over water depth provides a logarithmic velocity profile:

$$u(z) = \frac{u_*}{\kappa} \ln\left(\frac{z+d}{z_0}\right), \tag{3}$$

where u_* is the shear stress velocity; κ is the von Karman constant (not to be confused with the bottom friction coefficients κ_1, κ_2). The parameter z_0 may be related to the actual roughness:

$$z_0 = \frac{\kappa_N}{30},$$

where κ_N is the empirically determined roughness height.

For the depth-averaged model, the shear stress of the formation can be related to the depth-averaged velocity, through $\tau_b = c_{f_1} |U| U = u_*^2$. in combination with the logarithmic profile of equation (3), an expression for the coefficient of friction is found c_{f1}:

$$\frac{1}{\sqrt{c_{f_1}}} = \frac{1}{\kappa} \ln\left(e^{-1} \frac{d}{z_0}\right). \tag{4}$$

For a 3D model with a vertical dimension, the shear stress of the layer can be related to the coefficient of friction (c_{f2}) through $\tau_b = c_{f2} |u_b| u_b$. The bottom voltage is defined as:

$$\tau_b = |u_*| u_* \Rightarrow u_b = \frac{u_*}{\sqrt{c_{f_2}}}.$$

As a result, an expression for the coefficient of friction will be obtained:

$$\frac{1}{\sqrt{c_{f_2}}} = \frac{1}{\kappa} \ln\left(e^{-2} \frac{\Delta z_b}{z_0}\right). \tag{5}$$

The ratio between and is found by equating (4) and (5):

$$\frac{1}{\sqrt{c_{f_1}}} = \frac{1}{\sqrt{c_{f_2}}} - \frac{1}{\kappa} \ln\left(e^{-1} \frac{\Delta z_b}{d}\right).$$

Results. Calculations are performed for stationary and non-stationary (periodic flow) flows. In the stationary case, the gradient of the water level is constant over time. In the non-stationary case, a periodically changing flow is investigated. In both cases, numerical modelling is performed using linearized bottom friction corresponding to the analytical approach. The numerical response of horizontal (depth-averaged) velocities should correspond to analytical velocity profiles averaged over depth. The observed difference can only be caused by numerical approximations, i. e. time integration and (horizontal) discretization.

For both flow cases, the geometry of the computational domain is represented as a rectangular basin with two open borders on the short sides and a water depth of 12 m. The width of the pools is small (40 m) compared to the length. For the case of steady flow, the basin is extended by 20.000 m in length. A pool of this length is necessary for the full development of the water level gradient.

The boundary conditions for the stationary case determine the water level of 20 cm at the inflow boundary (left), the water level of 0 m at the outflow boundary (right) and zero normal velocity on the side walls and surface (upper boundary). Thus, the water level is fixed with a slope $i_w = 10^{-5}$ (Fig. 1).

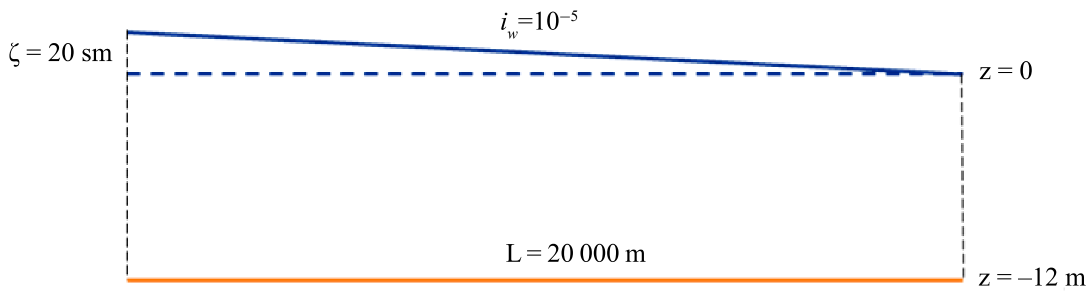

Fig. 1. Steady flow in a long channel

When the 3D results are averaged in depth (3D-DA), they can be directly compared with the corresponding results of the 2D model. The same values were selected for the roughness coefficient κ_N in both models, c_{f1} was chosen to be 0.002, κ_N should be 0.086 m, therefore, c_{f2} will be equal to 0.0042. This is the input data for a linearized bottom friction model.

Thus, the simulation with quadratic bottom friction was actually performed before the linearized case. This allows comparison between all models (2D and 3D, linear and quadratic).

In combination with quadratic bottom friction, two turbulence models are studied: the constant vortex viscosity and the Prandtl mixing length model. Ultimately, both calculations will result in the same velocity averaged over depth, provided that a certain value is selected for the viscosity of the vertical vortex corresponding to the selected specific bottom friction coefficients, resulting $v_t = 0.22$ m²/s (Table 1).

Table 1

Input parameters for the case of steady flow

Parameter	The calculated value of the parameter
c_{f1}	0.002
c_{f2}	0.004
κ_1	$2.9 \cdot 10^{-5}$
κ_2	$8.3 \cdot 10^{-5}$
v_t	0.22 m²/s

The simulation was performed for a long channel with linearized bottom friction. The values of the input parameters used for this simulation are summarized in Table 2. Theoretical velocity profiles for steady-state flow with linearized bottom friction are used to compare numerical results with analytical ones. Unlike analytical solutions, numerical modelling has minor deviations in the long run.

Table 2

Input parameters for stationary flow with linearized bottom friction

Fundamental parameters	Derived parameters
$i_\omega = 10^{-5}$	$\Delta z_b = d/nz = 2$ m
$d = 12$ m	0.004
$\kappa_N = 0.086$ m	$\kappa_1 = 5.7 \cdot 10^{-5}$
$nz = 6$	$\kappa_2 = 8.3 \cdot 10^{-5}$
v_t	$v_t = 0.22$ m²/s

In both 2D and 3D, numerical results are consistent with analytical solutions. When constructing the velocity profile $u(z)$ in AZOV3D, an ideal correspondence to the theoretical parabolic profile was demonstrated (Fig. 2, green shows the result of AZOV3D, black shows the analytical solution).

The following is an example that examines the effect of linearization on the numerical solution in comparison with analytical profiles for the same long channel. For this example, AZOV3D models are used that calculate bottom friction in a quadratic formulation.

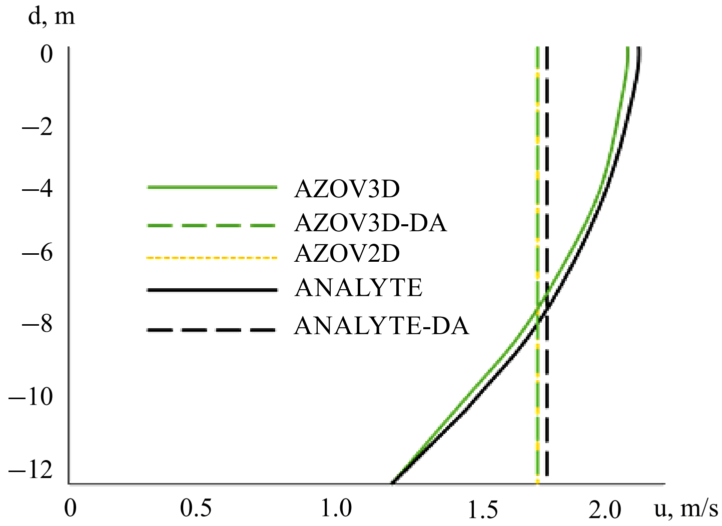

Fig. 2. Parabolic velocity profiles for steady flow with linearized bottom friction and constant vertical vortex viscosity

First, a simulation with a constant vortex viscosity is performed, and then another turbulence model is tested — the mixing length model. Table 3 shows the values of the input parameters used for this simulation.

Table 3

Input parameters for the stationary case with quadratic lower friction

Fundamental parameters	Derived parameters
$i_\omega = 10^{-5}$ $d = 12$ m $\kappa_N = 0.086$ m $nz = 6$ $\kappa = 0.4$	$\Delta z_b = d/nz = 2$ m $c_{f1} = 0.002$ $c_{f2} = 0.0042$ $v_t = 0.22$ m^2/s $u_* = 0.077$ m/s

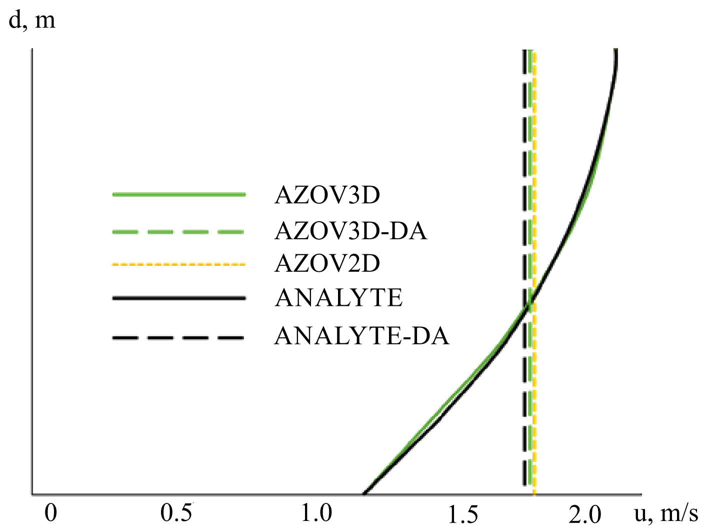

Fig. 3. Parabolic velocity profiles for steady-state flow with quadratic bottom friction and constant vertical vortex viscosity

The numerical results are compared with the theoretical velocity profile in a similar way to the linearized case and are completely consistent with the analytical solution, as shown in Fig. 3 for the case with constant vortex viscosity and in Fig. 4 for the case with the mixing length model. The velocity profiles are reproduced correctly: in the case of a constant vortex viscosity, a parabolic velocity profile, and in the case of a mixing length model, a logarithmic profile. Both 2D and 3D modeling correspond to the theory.

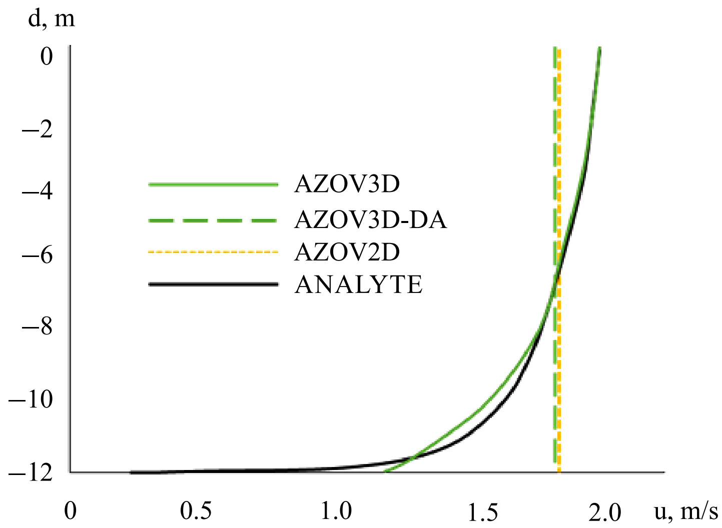

Fig. 4. Logarithmic velocity profiles for steady-state flow with quadratic bottom friction and viscosity determined by the mixing length model

For both the linearized formulation of bottom friction and the quadratic one, it is proved that the numerical results for stationary flow, as expected, show great similarity with analytical solutions. Since the surface height is much less than the depth of the water, advection can be neglected, so that the numerical characteristic is reproduced in full accordance with the theory, providing a good starting point for unsteady flow.

The numerical results for the unsteady flow show a good agreement with the theory. In addition, the analytical approach showed that the speed calculated using a 2D model is more likely to be greater than the 3D speed than vice versa. This behavior is certainly reflected in the numerical examples above, since all calculated ratios are greater than one.

Discussion and Conclusion. Calculations have been performed for stationary and non-stationary (periodic flow) currents using linearized bottom friction. In both 2D and 3D, numerical results are consistent with analytical solutions. When constructing the velocity profile in AZOV3D, an ideal correspondence to the theoretical parabolic profile is shown.

The effect of linearization on the numerical solution is studied in comparison with analytical profiles using models calculating bottom friction in a quadratic formulation. In combination with quadratic bottom friction, two turbulence models are studied: the constant vortex viscosity and the Prandtl mixing length model. The numerical results are compared with the theoretical velocity profile in a similar way to the linearized case and are consistent with the analytical solution, but unlike analytical solutions, numerical modelling has minor deviations in the long run. A parabolic velocity profile is obtained in the case of a constant vortex viscosity, and a logarithmic one in the case of a mixing length model.

For the linearized and quadratic formulations of bottom friction, it is proved that the numerical results for the case of stationary flow show great similarity with analytical solutions, since the surface height is much less than the water depth and advection can be neglected. The numerical results for the unsteady flow also show a good agreement with the theory. The analytical approach showed that the speed calculated using a 2D model is highly likely to be higher than the 3D speed, which is confirmed by numerical data. The study of flows in various turbulence models makes it possible to determine the influence of nonlinear effects on the accuracy of forecasts in vortex viscosity models and their applicability to various flow conditions.

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