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Finite Difference Method for Solving Parabolic-Type Integro-Differential Equations in Multidimensional Domain with Nonhomogeneous First-Order Boundary Conditions

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Abstract

Introduction. We investigate a multidimensional (in terms of spatial variables) parabolic-type integro-differential equation with nonhomogeneous first-order boundary conditions. The locally one-dimensional finite difference scheme developed herein can be applied to solve various applied problems leading to multidimensional parabolic-type integro-differential equations. Examples include mathematical modeling of cloud processes, addressing the issue of active intervention in convective clouds to prevent hail and artificially enhance precipitation, as well as describing the droplet mass distribution function due to microphysical processes such as condensation, coagulation, fragmentation, and freezing of droplets in convective clouds.

Materials and Methods. In this study, an approximate solution to the initial-boundary value problem is constructed using the locally one-dimensional scheme of A.A. Samarsky with a specified order of approximation $O(h^2+\tau)$. The primary research method employed is the method of energy inequalities.

Results. A priori estimates have been obtained in the discrete interpretation, from which uniqueness, stability, and convergence of the solution of the locally one-dimensional difference scheme to the solution of the original differential problem follow, with a convergence rate equal to the order of approximation of the difference scheme.

Discussion and Conclusions. The research findings can be utilized for further development of boundary value problem theory for parabolic equations with variable coefficients. Additionally, they may find applications in the fields of difference scheme theory, computational mathematics, and numerical modelling.

Keywords: multidimensional problem, diffusion equation, parabolic equation, first-order condition, difference schemes, locally one-dimensional scheme, a priori estimate, stability, convergence

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Научная статья

Разностный метод решения интегро-дифференциального уравнения параболического типа в многомерной области с неоднородными краевыми условиями первого рода

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Аннотация

Введение. Изучается многомерное (по пространственным переменным) интегро-дифференциальное уравнение параболического типа с неоднородными граничными условиями первого рода. Построенная локально-одномерная разностная схема может быть использована при решении прикладных задач, приводящих к многомерным интегро-дифференциальным уравнениям параболического типа, например, при математическом моделировании облачных процессов, при рассмотрении проблемы активного воздействия на конвективные облака с целью предотвращения града и искусственного увеличения осадков, а также при описании функции распределения по массам капель за счет микрофизических процессов конденсации, коагуляции, дробления и замерзания капель в конвективных облаках.

Материалы и методы. В данной работе для приближенного решения начально-краевой задачи построена локально-одномерная схема А.А. Самарского с порядком аппроксимации $O(h^2+\tau)$. Основной метод исследования — метод энергетических неравенств.

Результаты исследования. Получены априорные оценки в разностной трактовке, откуда следуют единственность, устойчивость, а также сходимость решения локально-одномерной разностной схемы к решению исходной дифференциальной задачи со скоростью, равной порядку аппроксимации разностной схемы.

Обсуждение и заключения. Результаты исследования могут быть использованы для дальнейшей разработки теории краевых задач для параболических уравнений с переменными коэффициентами, а также могут найти применение в области теории разностных схем, в области вычислительной математики и численного моделирования.

Ключевые слова: многомерная задача, уравнение диффузии, параболическое уравнение, условие первого рода, разностные схемы, локально-одномерная схема, априорная оценка, устойчивость, сходимость

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Introduction. Integro-differential equations, where the unknown function appears in the differential expression and, simultaneously, is present under the integral sign, are of great interest from the perspective of physical applications. There exists an extensive bibliography devoted to the subject of integro-differential equations. A detailed overview of advancements in this field up to 1962 is provided by M.M. Vainberg in [1]. The necessity of considering Volterra operator equations was first indicated by academician M.M. Lavrentiev in his report [2] at the International Congress of Mathematicians in Nice in 1970.

Various boundary value problems for ordinary integro-differential equations are addressed in works [3–4], while Sobolev-type integro-differential equations are studied in works [5–7]. Mathematical models accounting for memory effects in diffusion, arising in viscoelastic force models in non-Newtonian fluids and resulting from the modified Fourier's law applied to anisotropic heterogeneous media, are explored in [8–10]. Diffusion models featuring integral terms in the boundary flux are investigated in [11].

The objective of this study is to construct and investigate the locally one-dimensional difference scheme of A.A. Samarsky (LOD) of a specified approximation order $O(h^2+\tau)$ for the approximate solution of a boundary value problem with nonhomogeneous first-order boundary conditions for a multidimensional integro-differential parabolic equation.

The scientific novelty of this work lies in the development of the LOD and the derivation of a priori estimates in the discrete form for solving the LOD with nonhomogeneous first-order boundary conditions based on the method of energy inequalities. These estimates imply uniqueness of solution, continuous and uniform dependence of the solution on the input data, as well as convergence of the solution of the LOD to the solution of the original differential problem at a rate equal to the order of approximation of the difference scheme.

Methods for splitting multidimensional problems into one-dimensional ones are the subject of works by J. Douglas, D.W. Peaceman, H.H. Rachford [12–13], N.N. Yanenko [14], A.A. Samarsky [15], G.I. Marchuk [16], E.G. Dyakonov [17], and others.

Numerical methods for solving local and nonlocal boundary value problems for multidimensional partial differential equations of parabolic type based on various splitting methods are discussed in works by the author [18–20].

Materials and methods

Problem Statement. In a closed domain $\overline{Q}_T=\overline{G}\times[0,T]$, the base of which is a p-dimensional cube $\overline{G}=\{x=(x_1,x_2,...,x_p):0\le x_\alpha\le l,\ \ \alpha=1,2,...,p\}$ with boundary Γ, $\overline{G}=G\cup\Gamma$, the following problem is considered:

$$\frac{\partial u}{\partial t}+\int_0^t\rho_1(x,t,\tau)u(x,\tau)d\tau=Lu+f(x,t),\quad (x,t)\in Q_T, \tag{1}$$

$$u\big|_\Gamma=\mu(x,t),\quad 0\le t\le T, \tag{2}$$

$$u(x,0) = u_0(x), \quad x \in \overline{G}, \tag{3}$$

where

$$Lu = \sum_{\alpha=1}^{p} L_\alpha u, \; L_\alpha u = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial u}{\partial x_\alpha}\right) - q_\alpha(x,t)u - \int_0^l \rho_{2,\alpha}(\xi,t)u(\xi,t)d\xi_\alpha,$$

$$0 < c_0 \le k_\alpha(x,t) \le c_1, \quad \left|k_{\alpha,x_\alpha}(x,t)\right|, \; \left|\rho_1(x,t,\tau)\right|, \; \left|\rho_{2,\alpha}(x,t)\right|, \left|q_\alpha(x,t)\right| \le c_2,$$

$$u(x,t) \in C^{4,2}(\overline{Q_T}), \; k_\alpha(x,t) \in C^{3,1}(\overline{Q_T}), \; \rho_1(x,t,\tau), \; \rho_{2,\alpha}(x,t), \; q_\alpha(x,t), \; f(x,t) \in C^{2,1}(\overline{Q_T}), \; 0 \le \tau \le t, \tag{4}$$

$\mu(x, t)$, $u_0(x)$ are continuous functions; $\alpha = 1,2,...,p$, c_0, c_1, c_2 = const > 0, $C^{m,n}$ is the class of functions continuous together with their partial derivatives of order m with respect to x and n with respect to t, $Q_T = G\times(0,T]$.

The presence of an integral over time in the investigated differential equation is associated with the necessity to account for the dependence of instantaneous values of characteristics of the described object on their corresponding previous values, i. e., the influence of the system's past on its current state. In contemporary literature, such technical and natural systems are referred to as systems with delay, inheritance, or dynamic memory. The presence of a nonlocal source in integral form in the equation is entirely natural from physical considerations and arises in mathematical modeling in cases where there are sources (or sinks depending on the sign of $\rho_{2,\alpha}(\xi,t)$) and it is impossible to obtain information about the ongoing process through direct measurements or when it is only possible to measure certain averaged (integral) characteristics of the sought quantity.

For the approximate solution of the initial-boundary value problem, a locally one-dimensional scheme by A.A. Samarsky with an order of approximation $O(h^2+\tau)$ is constructed. The main research method employed is the method of energy inequalities. Two theorems are formulated and proven: a theorem on stability and a theorem on convergence.

Research Results

1. Construction of the Locally One-Dimensional Scheme (LOD). For each direction O_{x_α} we introduce a uniform grid with step size $h = \frac{l}{N}$ (a cubic grid with step h):

$$\overline{\omega}_h = \overline{\omega}^p_{h,\alpha}, \quad \overline{\omega}_{h,\alpha} = \left\{x_\alpha^{(i_\alpha)} = i_\alpha h : i_\alpha = 1,...,N-1, \quad x_\alpha^{(0)} = 0, \quad x_\alpha^{(N)} = N\frac{h}{2}\right\},$$

$$\omega_h = \omega^p_{h,\alpha}, \quad \omega_{h,\alpha} = \left\{x_\alpha^{(i_\alpha)} = i_\alpha h : i_\alpha = 1,...,N-1\right\}, \quad \alpha = 1,2,...,p.$$

On the interval $[0,T]$ we also introduce a uniform grid $\overline{\omega}_\tau = \{t_j = j\tau, \; j = 0,1,...,j_0\}$ with step size $\tau = T/j_0$. Each of the intervals $[t_j, t_{j+1}]$ is divided into p parts, introducing points $t_{j+\frac{\alpha}{p}} = t_j + \tau\frac{\alpha}{p}$, $\alpha = 1,2,..., p-1$, and denoting $\Delta_\alpha = \left(t_{j+\frac{\alpha-1}{p}}, t_{j+\frac{\alpha}{p}}\right]$ as the half-interval, where $\alpha = 1,2,..., p$.

Equation (1) is rewritten as

$$\sum_{\alpha=1}^{p} L_\alpha u = 0, \quad L_\alpha u = \frac{1}{p}\frac{\partial u}{\partial t} - \overline{L}_\alpha u - f_\alpha, \quad \sum_{\alpha=1}^{p} f_\alpha = f,$$

where $f_\alpha(x, t)$, ($\alpha = 1,2,..., p$) are arbitrary functions possessing the same smoothness as $f(x, t)$ satisfying the normalization condition $\sum_{\alpha=1}^{p} f_\alpha = f.$

For each half-interval Δ_α, ($\alpha = 1,2,..., p$), we will sequentially solve the problems

$$L_\alpha \vartheta_{(\alpha)} = \frac{1}{p}\frac{\partial \vartheta_{(\alpha)}}{\partial t} - \overline{L}_\alpha \vartheta_{(\alpha)} - f_\alpha, \quad x \in G, \quad t \in \Delta_\alpha, \quad \alpha = 1,2,...,p, \tag{5}$$

$$\begin{cases} \vartheta_\alpha = \mu_{-\alpha}, & x_\alpha = 0, \\ \vartheta_\alpha = \mu_{+\alpha}, & x_\alpha = l, \end{cases}$$

subject to the condition [21] that

$$\vartheta_{(1)}(x,0) = u_0(x), \quad \vartheta^j_{(1)}(x,t_j) = \vartheta^{j-1}_{(p)}(x,t_j), \quad j = 1,2,...,j_0,$$

$$\vartheta^j_{(\alpha)}(x,t_{j+\frac{\alpha-1}{p}}) = \vartheta^j_{(\alpha-1)}(x,t_{j+\frac{\alpha-1}{p}}), \quad \alpha = 2,....,p, \quad j = 1,2,...,j_0-1,$$

where $\overline{L}_\alpha \vartheta_{(\alpha)} = \frac{\partial}{\partial x_\alpha}\left(k_\alpha(x,t)\frac{\partial \vartheta_{(\alpha)}}{\partial x_\alpha}\right) - d_\alpha \vartheta_{(\alpha)} - \int_0^l \rho_{2,\alpha}(\xi,t)\vartheta_{(\alpha)}(\xi,t)d\xi_\alpha - \frac{1}{p}\int_0^t \rho_1(x,t,\tau)\vartheta_{(\alpha)}(x,\tau,t)d\tau,$

$\mu_{-\alpha} = \mu(x',0,t)$, $\mu_{+\alpha} = \mu(x',l,t)$ are continuous functions.

We approximate each equation (5) of index α with an implicit scheme on the half-interval Δ_α, then we obtain a chain of p one-dimensional difference schemes:

$$\frac{y^{j+\frac{\alpha}{p}} - y^{j+\frac{\alpha-1}{p}}}{\tau} = \Lambda_\alpha y^{j+\frac{\alpha}{p}} + \varphi_\alpha^{j+\frac{\alpha}{p}}, \quad x_\alpha \in \omega_h, \quad \alpha = 1,2,...,p, \tag{6}$$

$$\begin{cases} y^{j+\frac{\alpha}{p}} = \mu_{-\alpha}, & x_\alpha = 0, \\ y^{j+\frac{\alpha}{p}} = \mu_{+\alpha}, & x_\alpha = l, \end{cases} \tag{7}$$

$$y(x,0) = u_0(x), \tag{8}$$

where $\Lambda_\alpha y^{j+\frac{\alpha}{p}} = \left(a_\alpha y_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}}\right)_{x_\alpha} - d_\alpha y^{j+\frac{\alpha}{p}} - \sum_{i_\alpha=0}^{N} p_{2,\alpha} y_{i_\alpha}^{j+\frac{\alpha}{p}} \hbar - \frac{1}{p}\sum_{i_\alpha=0}^{N} p_1(x_1,x_2,....,x_p;t_j,t_{j'})y(x,t_{j'+\frac{\alpha}{p}})\tau,$

$$a_\alpha = k_\alpha(x^{(-0.5\alpha)},\bar{t}), \quad x^{(-0.5\alpha)} = (x_1,...,x_{\alpha-1},x_\alpha - 0.5h_\alpha, x_{\alpha+1},....,x_p), \qquad \hbar = \begin{cases} h, i_\alpha = \overline{1,N-1} \\ \frac{h}{2}, i_\alpha = 0, N, \end{cases}$$

$$d_\alpha = q_\alpha(x,\bar{t}), \quad \varphi_\alpha = f_\alpha(x,\bar{t}), \quad p_1(x,t,\bar{t}) = \rho_1(x,t,\bar{t}), \quad p_{2,\alpha} = \rho_{2,\alpha}(x,\bar{t}), \quad \bar{t} = t_{j+\frac{1}{2}}.$$

2. Approximation Error of the LOD. Substituting $y^{j+\frac{\alpha}{p}} = z^{j+\frac{\alpha}{p}} + u^{j+\frac{\alpha}{p}}$ into the scheme (6)–(8), we obtain a problem for the error $z^{j+\frac{\alpha}{p}}$:

$$\frac{z^{j+\frac{\alpha}{p}} - z^{j+\frac{\alpha-1}{p}}}{\tau} = \Lambda_\alpha z^{j+\frac{\alpha}{p}} + \psi_\alpha^{j+\frac{\alpha}{p}}, \tag{9}$$

$$z^{j+\frac{\alpha}{p}} = 0 \quad \text{by} \quad x \in \gamma_{h,\alpha}, \quad z(x,0) = 0,$$

where $u^{j+\frac{\alpha}{p}}$ is the solution of the original problem (1)–(3), and $y^{j+\frac{\alpha}{p}}$ is the solution of the difference problem (6)–(8),

$$\psi_\alpha^{j+\frac{\alpha}{p}} = \Lambda_\alpha u^{j+\frac{\alpha}{p}} + \varphi_\alpha^{j+\frac{\alpha}{p}} - \frac{u^{j+\frac{\alpha}{p}} - u^{j+\frac{\alpha-1}{p}}}{\tau}.$$

Denoting $\overset{0}{\psi}_\alpha = \left(\overline{L}_\alpha u + f_\alpha - \frac{1}{p}\frac{\partial u}{\partial t}\right)^{j+1/2}$ and noticing that $\sum_{\alpha=1}^{p}\overset{0}{\psi}_\alpha = 0,$ if $\sum_{\alpha=1}^{p} f_\alpha = f,$ is represented as a sum $\psi_\alpha^{j+\frac{\alpha}{p}} = \overset{0}{\psi}_\alpha + \psi_\alpha^*$:

$$\psi_\alpha^{j+\frac{\alpha}{p}} = \Lambda_\alpha u^{j+\frac{\alpha}{p}} + \varphi_\alpha^{j+\frac{\alpha}{p}} - \frac{u^{j+\frac{\alpha}{p}} - u^{j+\frac{\alpha-1}{p}}}{\tau} + \overset{0}{\psi}_\alpha - \overset{0}{\psi}_\alpha = \left(\Lambda_\alpha u^{j+\frac{\alpha}{p}} - \overline{L}_\alpha u^{j+\frac{1}{2}}\right) +$$

$$+\left(\varphi_\alpha^{j+\frac{\alpha}{p}} - f_\alpha^{j+\frac{1}{2}}\right) - \left(\frac{u^{j+\frac{\alpha}{p}} - u^{j+\frac{\alpha-1}{p}}}{\tau} - \frac{1}{p}\left(\frac{\partial u}{\partial t}\right)^{j+\frac{1}{2}}\right) + \overset{0}{\psi}_\alpha = \overset{0}{\psi}_\alpha + \psi_\alpha^*.$$

It is obvious that $\psi_\alpha^* = O(h^2+\tau)$, $\overset{0}{\psi}_\alpha = O(1)$, $\sum_{\alpha=1}^{p}\psi_\alpha^{j+\frac{\alpha}{p}} = \sum_{\alpha=1}^{p}\overset{0}{\psi}_\alpha + \sum_{\alpha=1}^{p}\psi_\alpha^* = O(h^2+\tau).$

3. Stability of the LOD. For the stability of the LOD, the following theorem holds.

Theorem 1. Suppose the smoothness and boundedness conditions (4) are satisfied. Then, the locally one-dimensional scheme (6)–(8) is stable with respect to the right-hand side and initial data. Consequently, for the solution of the scheme (6)–(8) with $\tau \le \tau_0$ the following estimate holds

$$|\rho_p y^{j+1}|^2_{L_2(\omega_h)} + \sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(\|[\rho_\alpha y_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)} + |[y^{j'+\frac{\alpha}{p}}]|^2_{L_2(\bar{\omega}_h)} + \right) \le \tag{10}$$

$$\le M\left(\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}|[\varphi^{j'+\frac{\alpha}{p}}]|^2_{L_2(\bar{\omega}_h)} + \frac{\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\sum_{i_\beta\ne i_\alpha}\left(\mu^2_{-\alpha}(0,x',t_{j'}) + \mu^2_{+\alpha}(l,x',t_{j'})\right)\overline{H}}{\hbar} + |[y^0]|^2_{L_2(\bar{\omega}_h)}\right).$$

where $M = \text{const} > 0$ does not depend on h and τ.

Here M_i, $i = 1,2,...$, denotes positive constants depending solely on the input data of the problem (1)–(3).

Proof. he analysis is conducted using the method of energy inequalities, for which we introduce scalar products and norms in the following form:

$$\frac{1}{p}y_{\bar{t}}^{(\alpha)} = \frac{y^{j+\frac{\alpha}{p}} - y^{j+\frac{\alpha-1}{p}}}{\tau},\ (u,v)_\alpha = \sum_{i_\alpha=1}^{N-1}u_{i_\alpha}v_{i_\alpha}h,\ \|y^{(\alpha)}|^2_{L_2(\alpha)} = \sum_{i_\alpha=1}^{N-1}y^2_{i_\alpha}h,$$

$$(u,v) = \sum_{x\in\omega_h}uvH,\quad H = h^p,\ |y^{(\alpha)}|^2_{L_2(\omega_h)} = \sum_{i_\beta\ne i_\alpha}|y^{(\alpha)}|^2_{L_2(\alpha)}H/h,\quad [u,v]_\alpha = \sum_{i_\alpha=0}^{N}u_{i_\alpha}v_{i_\alpha}\hbar,\ |[y^{(\alpha)}]|^2_{L_2(\alpha)} = \sum_{i_\alpha=0}^{N}y^2_{i_\alpha}\hbar,$$

$$[u,v] = \sum_{x\in\bar{\omega}_h}uv\overline{H},\ \overline{H} = \hbar^p,\ |[y^{(\alpha)}]|^2_{L_2(\bar{\omega}_h)} = \sum_{i_\beta\ne i_\alpha}|[y^{(\alpha)}]|^2_{L_2(\alpha)}\overline{H}/\hbar.$$

Multiplying equation (6) by $y^{(\alpha)}h$, where $y^{(\alpha)} = y^{j+\frac{\alpha}{p}}$ summing over s_α from η_α to ξ_α, we obtain [22]:

$$\frac{1}{p}\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}y^{(\alpha)}_{\bar{t},s_\alpha}y^{(\alpha)}_{s_\alpha}h = \sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\Lambda_\alpha y^{(\alpha)}_{s_\alpha}y^{(\alpha)}_{s_\alpha}h + \sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\varphi_{(\alpha),s_\alpha}y^{(\alpha)}_{s_\alpha}h,\quad 0\le\eta_\alpha\le\xi_\alpha\le N. \tag{11}$$

We transform each term of identity (11):

$$\frac{1}{p}\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}y^{(\alpha)}_{\bar{t},s_\alpha}y^{(\alpha)}_{s_\alpha}h = \frac{1}{2p}\left(\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y^{(\alpha)}_{s_\alpha}\right)^2h\right)_{\bar{t}} + \frac{\tau}{2p}\left(\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y^{(\alpha)}_{\bar{t},s_\alpha}\right)^2h\right), \tag{12}$$

$$\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\Lambda_\alpha y^{(\alpha)}_{s_\alpha}y^{(\alpha)}_{s_\alpha}h = \sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}(a_\alpha y_{\bar{x}_\alpha,s_\alpha})_{x_\alpha}y^{(\alpha)}_{s_\alpha}h - \sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}d_\alpha\left(y^{(\alpha)}_{s_\alpha}\right)^2h - \sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y^{(\alpha)}_{s_\alpha}\sum_{i_\alpha=0}^{N}p_{2,\alpha,i_\alpha}y_{i_\alpha}^{j+\frac{\alpha}{p}}\hbar\right)h - \tag{13}$$

$$-\frac{1}{p}\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y^{(\alpha)}_{s_\alpha}\sum_{j'=0}^{j}p_1(x,t_j,t_{j'})y\left(x,t^{j'+\frac{\alpha}{p}}\right)\tau\right)h = -\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha+1}a_{\alpha,s_\alpha}\left(y_{\bar{x}_\alpha,s_\alpha}\right)^2h + a_{\alpha,\xi_\alpha+1}y^{(\alpha)}_{\bar{x}_\alpha,\xi_\alpha+1}y^{(\alpha)}_{\xi_\alpha+1} - a_{\alpha,\eta_\alpha}y^{(\alpha)}_{\bar{x}_\alpha,\eta_\alpha}y^{(\alpha)}_{\eta_\alpha-1} -$$

$$-\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}d_{\alpha,s_\alpha}\left(y^{(\alpha)}_{s_\alpha}\right)^2h - \sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y^{(\alpha)}_{s_\alpha}\sum_{i_\alpha=0}^{N}p_{2,\alpha,i_\alpha}y_{i_\alpha}^{j+\frac{\alpha}{p}}\hbar\right)h - \frac{1}{p}\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y^{(\alpha)}_{s_\alpha}\sum_{j'=0}^{j}p_1(x,t_j,t_{j'})y\left(x,t^{j'+\frac{\alpha}{p}}\right)\tau\right)h,$$

$$\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\varphi_{(\alpha),s_\alpha}y^{(\alpha)}_{s_\alpha}h \le \frac{1}{2}\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\varphi^2_{(\alpha),s_\alpha}h + \frac{1}{2}\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y^{(\alpha)}_{s_\alpha}\right)^2h. \tag{14}$$

Let's transform the expression $a_{\alpha,\xi_\alpha+1}y^{(\alpha)}_{\bar{x}_\alpha,\xi_\alpha+1}y^{(\alpha)}_{\xi_\alpha+1} - a_{\alpha,\eta_\alpha}y^{(\alpha)}_{\bar{x}_\alpha,\eta_\alpha}y^{(\alpha)}_{\eta_\alpha-1}$ separately. Then, taking into account $(y^2)_{(\bar{x}_\alpha)} = 2y_{(\bar{x}_\alpha)}y^{(\alpha)} - hy^2_{(\bar{x}_\alpha)}$ we get:

$$a_{\alpha,\xi_\alpha+1}y^{(\alpha)}_{\bar{x}_\alpha,\xi_\alpha+1}y^{(\alpha)}_{\xi_\alpha+1} - a_{\alpha,\eta_\alpha}y^{(\alpha)}_{\bar{x}_\alpha,\eta_\alpha}y^{(\alpha)}_{\eta_\alpha-1} = \frac{1}{2}\left(a_\alpha y^2\right)_{\bar{x}_\alpha,\xi_\alpha+1} - \frac{1}{2}\left(a_\alpha\right)_{\bar{x}_\alpha,\xi_\alpha+1}y^2_{\xi_\alpha} + \tag{15}$$

$$+\frac{1}{2}a_{\xi_\alpha+1}y^2_{\bar{x}_\alpha,\xi_\alpha+1}h - \frac{1}{2}\left(a_\alpha y^2\right)_{\bar{x}_\alpha,\eta_\alpha} + \frac{1}{2}\left(a_\alpha\right)_{\bar{x}_\alpha,\eta_\alpha}y^2_{\eta_\alpha-1} + \frac{1}{2}a_{\eta_\alpha}y^2_{\bar{x}_\alpha,\eta_\alpha}h.$$

Considering transformations (12)–(15), from (11) we obtain:

$$\frac{1}{2p}\left(\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y^{(\alpha)}_{s_\alpha}\right)^2h\right)_{\bar{t}} + \frac{\tau}{2p}\left(\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y^{(\alpha)}_{\bar{t},s_\alpha}\right)^2h\right) + \frac{1}{2}\sum_{s_\alpha=\eta_\alpha-1}^{\xi_\alpha}a_{\alpha,s_\alpha}\left(y_{\bar{x}_\alpha,s_\alpha}\right)^2h \le \tag{16}$$

$$\le\frac{1}{2}\left(a_\alpha y^2\right)_{\bar{x}_\alpha,\xi_\alpha+1} - \frac{1}{2}\left(a_\alpha\right)_{\bar{x}_\alpha,\xi_\alpha+1}y^2_{\xi_\alpha} + \frac{1}{2}\left(a_\alpha\right)_{\bar{x}_\alpha,\eta_\alpha}y^2_{\eta_\alpha-1} - \frac{1}{2}\left(a_\alpha y^2\right)_{\bar{x}_\alpha,\eta_\alpha} -$$

$$-\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y_{s_\alpha}^{(\alpha)}\sum_{i_\alpha=0}^{N}p_{2,\alpha,i_\alpha}y_{i_\alpha}^{j+\frac{\alpha}{p}}\hbar\right)h-\frac{1}{p}\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y_{s_\alpha}^{(\alpha)}\sum_{j'=0}^{j}p_1\left(x,t_j,t_{j'}\right)y(x,t^{j'+\frac{\alpha}{p}})\tau\right)h-$$

$$-\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}d_{\alpha,s_\alpha}\left(y_{s_\alpha}^{(\alpha)}\right)^2h+\frac{1}{2}\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\varphi_{(\alpha),s_\alpha}^2h+\frac{1}{2}\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y_{s_\alpha}^{(\alpha)}\right)^2h.$$

Now, let's multiply (16) by h and sum over ξ_α from η_α to N, then multiply the resulting inequality by h and sum over η_α from 0 to N. Then we get:

$$\frac{1}{2p}\left(\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}h\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y_{s_\alpha}^{(\alpha)}\right)^2h\right)_{\bar t}+\frac{1}{2}\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}h\sum_{s_\alpha=\eta_\alpha-1}^{\xi_\alpha}a_{\alpha,s_\alpha}\left(y_{\bar x_\alpha,s_\alpha}\right)^2h\le \tag{17}$$

$$\le\frac{1}{2}\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}\left(a_\alpha y^2\right)_{\bar x_\alpha,\xi_\alpha+1}h-\frac{1}{2}\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}\left(a_\alpha\right)_{\bar x_\alpha,\xi_\alpha+1}y_{\xi_\alpha}^2h+$$

$$+\frac{1}{2}\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}\left(a_\alpha\right)_{\bar x_\alpha,\eta_\alpha}y_{\eta_\alpha-1}^2h-\frac{1}{2}\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}\left(a_\alpha y^2\right)_{\bar x_\alpha,\eta_\alpha}h-\frac{1}{2}\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}h\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y_{s_\alpha}^{(\alpha)}\sum_{i_\alpha=0}^{N}p_{2,\alpha,i_\alpha}y_{i_\alpha}^{j+\frac{\alpha}{p}}\hbar\right)h-$$

$$-\frac{1}{p}\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}h\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y_{s_\alpha}^{(\alpha)}\sum_{j'=0}^{j}p_1\left(x,t_j,t_{j'}\right)y(x,t^{j'+\frac{\alpha}{p}})\tau\right)h+\frac{1}{2}\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}h\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\varphi_{(\alpha),s_\alpha}^2h+M_1\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}h\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\left(y_{s_\alpha}^{(\alpha)}\right)^2h.$$

We transform the sum $\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}h\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\varphi_{(\alpha),s_\alpha}^2h$ as follows:

$$\sum_{\eta_\alpha=0}^{N}h\sum_{\xi_\alpha=\eta_\alpha}^{N}h\sum_{s_\alpha=\eta_\alpha}^{\xi_\alpha}\varphi_{(\alpha),s_\alpha}^2h=\sum_{\eta_\alpha=0}^{N}h\sum_{s_\alpha=\eta_\alpha}^{N}\varphi_{(\alpha),s_\alpha}^2h\sum_{\xi_\alpha=s_\alpha}^{N}h=$$

$$=\sum_{\eta_\alpha=0}^{N}h\sum_{s_\alpha=\eta_\alpha}^{N}\left(l-x_\alpha\right)\varphi_{(\alpha),s_\alpha}^2h=\sum_{s_\alpha=0}^{N}\left(l-x_\alpha\right)\varphi_{(\alpha),s_\alpha}^2h\sum_{\eta_\alpha=0}^{s_\alpha}h=\sum_{s_\alpha=0}^{N}x_\alpha\left(l-x_\alpha\right)\varphi_{(\alpha),s_\alpha}^2h=\sum_{s_\alpha=1}^{N-1}x_\alpha\left(l-x_\alpha\right)\varphi_{(\alpha),s_\alpha}^2h.$$

Taking into account the last equation from (17), we find:

$$\frac{1}{2p}\left(\sum_{s_\alpha=1}^{N-1}x_\alpha\left(l-x_\alpha\right)\left(y_{s_\alpha}^{(\alpha)}\right)^2h\right)_{\bar t}+\frac{1}{2}\sum_{s_\alpha=1}^{N-1}x_\alpha\left(l-x_\alpha\right)a_{\alpha,s_\alpha}\left(y_{\bar x_\alpha,s_\alpha}\right)^2h\le \tag{18}$$

$$\le\frac{1}{2}\sum_{s_\alpha=0}^{N-1}x_\alpha\left(a_\alpha y^2\right)_{\bar x_\alpha,s_\alpha+1}h-\frac{1}{2}\sum_{s_\alpha=0}^{N-1}x_\alpha\left(a_\alpha\right)_{\bar x_\alpha,s_\alpha+1}y_{s_\alpha}^2h-\frac{1}{2}\sum_{s_\alpha=1}^{N}\left(l-x_\alpha\right)\left(a_\alpha y^2\right)_{\bar x_\alpha,s_\alpha}h+$$

$$+\frac{1}{2}\sum_{s_\alpha=1}^{N}\left(l-x_\alpha\right)\left(a_\alpha\right)_{\bar x_\alpha,s_\alpha}y_{s_\alpha-1}^2h-\frac{1}{2}\sum_{s_\alpha=1}^{N-1}x_\alpha\left(l-x_\alpha\right)\left(y_{s_\alpha}^{(\alpha)}\sum_{i_\alpha=0}^{N}p_{2,\alpha,i_\alpha}y_{i_\alpha}^{j+\frac{\alpha}{p}}\hbar\right)h-$$

$$-\frac{1}{p}\sum_{s_\alpha=1}^{N-1}x_\alpha\left(l-x_\alpha\right)\left(y_{s_\alpha}^{(\alpha)}\sum_{j'=0}^{j}p_1\left(x,t_j,t_{j'}\right)y\left(x,t^{j'+\frac{\alpha}{p}}\right)\tau\right)h+\frac{1}{2}\sum_{s_\alpha=1}^{N-1}x_\alpha\left(l-x_\alpha\right)\varphi_{(\alpha),s_\alpha}^2h+M_1\sum_{s_\alpha=1}^{N-1}x_\alpha\left(l-x_\alpha\right)\left(y_{s_\alpha}^{(\alpha)}\right)^2h.$$

Now, let's estimate the first, second, and third terms of the right-hand side of (18):

$$\frac{1}{2}\sum_{s_\alpha=0}^{N-1}x_\alpha\left(a_\alpha y^2\right)_{\bar x_\alpha,s_\alpha+1}h-\frac{1}{2}\sum_{s_\alpha=0}^{N-1}x_\alpha\left(a_\alpha\right)_{\bar x_\alpha,s_\alpha+1}y_{s_\alpha}^2h-\frac{1}{2}\sum_{s_\alpha=1}^{N}\left(l-x_\alpha\right)\left(a_\alpha y^2\right)_{\bar x_\alpha,s_\alpha}h+ \tag{19}$$

$$+\frac{1}{2}\sum_{s_\alpha=1}^{N}\left(l-x_\alpha\right)\left(a_\alpha\right)_{\bar x_\alpha,s_\alpha}y_{s_\alpha-1}^2h=\frac{1}{2}\sum_{s_\alpha=1}^{N}\left(2x_\alpha-h-l\right)\left(a_\alpha y^2\right)_{\bar x_\alpha,s_\alpha}h-$$

$$-\frac{1}{2}\sum_{s_\alpha=0}^{N-1}\left(2x_\alpha+h-l\right)\left(a_\alpha\right)_{\bar x_\alpha,s_\alpha+1}y_{s_\alpha}^2h=\frac{1}{2}\sum_{s_\alpha=1}^{N}\left(\left(2x_\alpha-h-l\right)a_\alpha y^2\right)_{\bar x_\alpha,s_\alpha}h-$$

$$-\frac{1}{2}\sum_{s_\alpha=0}^{N-1}(2x_\alpha+h-l)_{\bar{x}_\alpha,s_\alpha}a_\alpha y_{s_\alpha}^2h-\frac{1}{2}\sum_{s_\alpha=0}^{N-1}(2x_\alpha+h-l)(a_\alpha)_{\bar{x}_\alpha,s_\alpha+1}y_{s_\alpha}^2h=$$

$$=-\sum_{s_\alpha=0}^{N}a_\alpha y_{s_\alpha}^2h-\frac{1}{2}\sum_{s_\alpha=0}^{N-1}(2x_\alpha+h-l)(a_\alpha)_{\bar{x}_\alpha,s_\alpha+1}y_{s_\alpha}^2h+\frac{l}{2}\left(a_{\alpha,N_\alpha}y_{N_\alpha}^2+a_{\alpha,0}y_0^2\right).$$

Taking into account the boundary conditions (2) and (19), we obtain from (18):

$$\frac{1}{2p}\left(\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{s_\alpha}^{(\alpha)}\right)^2h\right)_{\bar{t}}+\frac{c_0}{2}\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{\bar{x}_\alpha,s_\alpha}\right)^2h+ \tag{20}$$

$$+c_0\sum_{s_\alpha=0}^{N}(y_{s_\alpha}^2)h\le-\frac{1}{2}\sum_{s_\alpha=0}^{N-1}(2x_\alpha+h-l)(a_\alpha)_{\bar{x}_\alpha,s_\alpha+1}y_{s_\alpha}^2h-\frac{1}{2}\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{s_\alpha}^{(\alpha)}\sum_{i_\alpha=0}^{N}p_{2,\alpha,i_\alpha}y_{i_\alpha}^{j+\frac{\alpha}{p}}\hbar\right)h-$$

$$-\frac{1}{p}\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{s_\alpha}^{(\alpha)}\sum_{j'=0}^{j}p_1(x,t_j,t_{j'})y\left(x,t^{j'+\frac{\alpha}{p}}\right)\tau\right)h+M_1\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{s_\alpha}^{(\alpha)}\right)^2h+M_2\left(\sum_{s_\alpha=1}^{N-1}\varphi_{(\alpha),s_\alpha}^2h+\mu_{-\alpha}^2+\mu_{+\alpha}^2\right).$$

Now, let's estimate the first, second, and third terms of the right-hand side of (20) using the Cauchy inequality with ε, Lemma 2 [22], and obtain:

$$-\sum_{s_\alpha=0}^{N-1}(2x_\alpha+h-l)(a_\alpha)_{\bar{x}_\alpha,s_\alpha+1}y_{s_\alpha}^2h\le M_1\,|[y^{(\alpha)}]|_{L_2(\alpha)}^2\le\varepsilon M_1\,|[\rho_\alpha y_{\bar{x}_\alpha}^{(\alpha)}]|_{L_2(\alpha)}^2+M_2^\varepsilon\,|\,\rho_\alpha y^{(\alpha)}\,|_{L_2(\alpha)}^2, \tag{21}$$

$$-\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{s_\alpha}^{(\alpha)}\sum_{i_\alpha=0}^{N}p_{2,\alpha,i_\alpha}y_{i_\alpha}^{(\alpha)}\hbar\right)h\le\frac{1}{4\varepsilon_1}\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{s_\alpha}^{(\alpha)}\right)^2h+\varepsilon_1\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(\sum_{i_\alpha=0}^{N}p_{2,\alpha,i_\alpha}y_{i_\alpha}^{(\alpha)}\hbar\right)^2h\le \tag{22}$$

$$\le\frac{1}{4\varepsilon_1}\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{s_\alpha}^{(\alpha)}\right)^2h+\varepsilon_1M_3\sum_{s_\alpha=1}^{N-1}\left(x_\alpha(l-x_\alpha)\sum_{i_\alpha=0}^{N}\left(y_{i_\alpha}^{(\alpha)}\right)^2\hbar\right)h\le$$

$$\le\frac{1}{4\varepsilon_1}\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{s_\alpha}^{(\alpha)}\right)^2h+\varepsilon_1M_4\sum_{s_\alpha=0}^{N}y_{s_\alpha}^2h\le\frac{1}{4\varepsilon_1}\,|\,\rho y^{(\alpha)}\,|_{L_2(\alpha)}^2+\varepsilon_1M_4\,|[y^{(\alpha)}]|_{L_2(\alpha)}^2,$$

$$-\frac{1}{p}\sum_{s_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(y_{s_\alpha}^{(\alpha)}\sum_{j'=0}^{j}p_1(x,t_j,t_{j'})y(x,t^{j'+\frac{\alpha}{p}})\tau\right)h\le \tag{23}$$

$$\frac{1}{2}\frac{1}{p^2}\sum_{i_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(\sum_{j'=0}^{j}p_1(x_{i_\alpha},t_j,t_{j'})y(x_{i_\alpha},t^{j'+\frac{\alpha}{p}})\tau\right)^2h+\frac{1}{2}\,|\,\rho_\alpha y^{j+\frac{\alpha}{p}}\,|_{L_2(\alpha)}^2\le$$

$$\le\frac{1}{2}\frac{1}{p^2}\sum_{i_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)\left(\sum_{j'=0}^{j}p_1^2(x_{i_\alpha},t_j,t_{j'})\tau\sum_{j'=0}^{j}y^2(x_{i_\alpha},t^{j'+\frac{\alpha}{p}})\tau\right)h+$$

$$+\frac{1}{2}\,|\,\rho_\alpha y^{j+\frac{\alpha}{p}}\,|_{L_2(\alpha)}^2\le M_6\sum_{j'=0}^{j}\tau\sum_{i_\alpha=1}^{N-1}x_\alpha(l-x_\alpha)y^2(x_{i_\alpha},t^{j'+\frac{\alpha}{p}})h+\frac{1}{2}\,|\,\rho_\alpha y^{j+\frac{\alpha}{p}}\,|_{L_2(\alpha)}^2=$$

$$=M_6\sum_{j'=0}^{j}\tau\,|\,\rho_\alpha y(x_{i_\alpha},t^{j'+\frac{\alpha}{p}})\,|_{L_2(\alpha)}^2+\frac{1}{2}\,|\,\rho_\alpha y^{j+\frac{\alpha}{p}}\,|_{L_2(\alpha)}^2,$$

where $\rho_\alpha=\sqrt{x_\alpha(l-x_\alpha)},\quad \rho_1=\rho_2=...=\rho_p$.

Considering (21)–(23), from (20) we find:

$$\frac{1}{p}\left(|\,\rho_\alpha y^{(\alpha)}\,|_{L_2(\alpha)}^2\right)_{\bar{t}}+|[\rho_\alpha y_{\bar{x}_\alpha}^{(\alpha)}]|_{L_2(\alpha)}^2+|[y^{(\alpha)}]|_{L_2(\alpha)}^2\le\varepsilon M_7\,|[\rho_\alpha y_{\bar{x}_\alpha}^{(\alpha)}]|_{L_2(\alpha)}^2+\varepsilon_1M_8\,|[y^{(\alpha)}]|_{L_2(\alpha)}^2+M_9\,|\,\rho_\alpha y^{(\alpha)}\,|_{L_2(\alpha)}^2+ \tag{24}$$

$$+M_{10}\sum_{j'=0}^{j}|\,\rho_\alpha y^{j'+\frac{\alpha}{p}}\,|_{L_2(\alpha)}^2\,\tau+M_{11}\left(|[\varphi_{(\alpha)}]|_{L_2(\alpha)}^2+\mu_{-\alpha}^2+\mu_{+\alpha}^2\right).$$

where $\|\cdot\|_{L_2(\alpha)}$ means that the norm is taken over the variable x_α with fixed values of other variables.

Choosing $\varepsilon=\dfrac{1}{2M_7},\ \varepsilon_1=\dfrac{1}{2M_8}$ from (24), we find:

$$\frac{1}{p}\left(\|\rho_\alpha y^{(\alpha)}\|^2_{L_2(\alpha)}\right)_{\bar t}+\|[\rho_\alpha y^{(\alpha)}_{\bar x_\alpha}]\|^2_{L_2(\alpha)}+|[y^{(\alpha)}]|^2_{L_2(\alpha)}\le \tag{25}$$

$$\le M_{12}|\rho_\alpha y^{(\alpha)}|^2_{L_2(\alpha)}+M_{13}\sum_{j'=0}^{j}\|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\alpha)}\tau+M_{14}\left(|[\varphi_{(\alpha)}]|^2_{L_2(\alpha)}+\mu^2_{-\alpha}+\mu^2_{+\alpha}\right).$$

Substituting the obtained estimates after summation over $i_\alpha\ne i_\beta,\ \beta=1,2,...,p$ into identity (25), we get the inequality:

$$\frac{1}{p}\left(|\rho_\alpha y^{(\alpha)}|^2_{L_2(\omega_h)}\right)_{\bar t}+\|[\rho_\alpha y_{\bar x_\alpha}]\|^2_{L_2(\omega_h)}+|[y^{(\alpha)}]|^2_{L_2(\bar\omega_h)}\le M_{15}|\rho_\alpha y^{(\alpha)}|^2_{L_2(\omega_h)}+ \tag{26}$$

$$+M_{16}\sum_{j'=0}^{j}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\tau+M_{17}\left(|[\varphi^{j+\frac{\alpha}{p}}]|^2_{L_2(\bar\omega_h)}+\sum_{i_\beta\ne i_\alpha}\left(\mu^2_{-\alpha}(t_j)+\mu^2_{+\alpha}(t_j)\right)\overline{H}/\hbar\right).$$

Summing (26) first over $\alpha=1,2,...,p$, and then multiplying both sides by τ and summing over j' from 0 to j, we obtain:

$$|\rho_p y^{j+1}|^2_{L_2(\omega_h)}+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(\|[\rho_\alpha y_{\bar x_\alpha}^{j'+\frac{\alpha}{p}}]\|^2_{L_2(\omega_h)}+|[y^{j'+\frac{\alpha}{p}}]|^2_{L_2(\bar\omega_h)}\right)\le \tag{27}$$

$$\le M_{18}\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}+M_{19}\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\sum_{s=0}^{j'}|\rho_\alpha y^{s+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\tau+$$

$$+M_{20}\left(\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(|[\varphi^{j'+\frac{\alpha}{p}}]|^2_{L_2(\bar\omega_h)}+\sum_{i_\beta\ne i_\alpha}\left(\mu^2_{-\alpha}(t_{j'})+\mu^2_{+\alpha}(t_{j'})\right)\overline{H}/\hbar\right)+|[y^0]|^2_{L_2(\bar\omega_h)}\right).$$

From (27), we have:

$$|\rho_p y^{j+1}|^2_{L_2(\omega_h)}\le M_{21}\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}+M_{20}F^j, \tag{28}$$

where $M_{21}=M_{18}+TM_{19}$,

$$F^j=\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(|[\varphi^{j'+\frac{\alpha}{p}}]|^2_{L_2(\bar\omega_h)}+\sum_{i_\beta\ne i_\alpha}\left(\mu^2_{-\alpha}+\mu^2_{+\alpha}\right)\overline{H}/\hbar\right)+|[y^0]|^2_{L_2(\bar\omega_h)}.$$

Now, let's show that the inequality holds:

$$\max_{1\le\alpha\le p}|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\le \nu_1\sum_{j'=0}^{j-1}\max_{1\le\alpha\le p}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}+\nu_2F^j,$$

where ν_1,ν_2 are known positive constants.

For this, let's rewrite inequality (26) as follows:

$$\frac{1}{p}|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\le\frac{1}{p}|\rho_\alpha y^{j+\frac{\alpha-1}{p}}|^2_{L_2(\omega_h)}+\tau M_{15}|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}+\tau M_{16}\sum_{j'=0}^{j}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\tau+ \tag{29}$$

$$+\tau M_{17}\left(|[\varphi^{j+\frac{\alpha}{p}}]|^2_{L_2(\bar\omega_h)}+\sum_{i_\beta\ne i_\alpha}\left(\mu^2_{-\alpha}+\mu^2_{+\alpha}\right)\overline{H}/\hbar\right).$$

Summing (29) over α' from 1 to α we obtain:

$$\frac{1}{p}\sum_{\alpha'=1}^{\alpha}|\rho_{\alpha'}y^{j+\frac{\alpha'}{p}}|^2_{L_2(\omega_h)}\le\frac{1}{p}\sum_{\alpha'=1}^{\alpha}|\rho_{\alpha'}y^{j+\frac{\alpha'-1}{p}}|^2_{L_2(\omega_h)}+$$

$$+\tau M_{15}\sum_{\alpha'=1}^{\alpha}|\rho_{\alpha'}y^{j+\frac{\alpha'}{p}}|^2_{L_2(\omega_h)}+\tau M_{16}\sum_{\alpha'=1}^{\alpha}\sum_{j'=0}^{j}|\rho_{\alpha'}y^{j'+\frac{\alpha'}{p}}|^2_{L_2(\omega_h)}\tau+$$

$$+\tau M_{17}\sum_{\alpha'=1}^{\alpha}\left(|[\varphi^{j+\frac{\alpha'}{p}}]|^2_{L_2(\bar{\omega}_h)}+\sum_{i_\beta\neq i_{\alpha'}}\left(\mu^2_{-\alpha'}+\mu^2_{+\alpha'}\right)\overline{H}/\hbar\right).$$

From the latter, we obtain:

$$|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\leq|\rho_1 y^j|^2_{L_2(\omega_h)}+\tau M_{15}\sum_{\alpha=1}^{p}|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}+\tau M_{16}\sum_{\alpha=1}^{p}\sum_{j'=0}^{j}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\tau+ \tag{30}$$

$$+\tau M_{17}\sum_{\alpha=1}^{p}\left(|[\varphi^{j+\frac{\alpha}{p}}]|^2_{L_2(\bar{\omega}_h)}+\sum_{i_\beta\neq i_\alpha}\left(\mu^2_{-\alpha}+\mu^2_{+\alpha}\right)\overline{H}/\hbar\right).$$

Without loss of generality, we can assume that

$$\max_{1\leq\alpha'\leq p}|\rho_{\alpha'}y^{j+\frac{\alpha'}{p}}|^2_{L_2(\omega_h)}=|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)},$$

otherwise, if (29) does not hold, we sum until α such that $|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}$ reaches its maximum value for a fixed j. Then, we rewrite (30) as

$$\max_{1\leq\alpha\leq p}|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\leq|\rho_1 y^j|^2_{L_2(\omega_h)}+p\tau M_{15}\max_{1\leq\alpha\leq p}|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}+pM_{16}T\sum_{j'=0}^{j}\max_{1\leq\alpha\leq p}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\tau+ \tag{31}$$

$$+\tau M_{17}\sum_{\alpha=1}^{p}\left(|[\varphi^{j+\frac{\alpha}{p}}]|^2_{L_2(\bar{\omega}_h)}+\sum_{i_\beta\neq i_\alpha}\left(\mu^2_{-\alpha}+\mu^2_{+\alpha}\right)\overline{H}/\hbar\right).$$

Since it follows from (28) that

$$|\rho_1 y^j|^2_{L_2(\omega_h)}=|\rho_p y^j|^2_{L_2(\omega_h)}\leq M_{21}\sum_{j'=0}^{j-1}\max_{1\leq\alpha\leq p}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}+M_{20}F^j,$$

from (31) we have

$$\left(1-\tau M_{22}\right)\max_{1\leq\alpha\leq p}|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\leq M_{23}\sum_{j'=0}^{j-1}\max_{1\leq\alpha\leq p}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}+M_{24}F^j, \tag{32}$$

where $M_{22}=pM_{15}+pM_{16}T,\ M_{23}=M_{21}+pM_{16}T.$

Choosing $\tau\leq\tau_0=\dfrac{1}{2M_{22}}$, from (32), we find

$$\max_{1\leq\alpha\leq p}|\rho_\alpha y^{j+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}\leq\nu_1\sum_{j'=0}^{j-1}\max_{1\leq\alpha\leq p}|\rho_\alpha y^{j'+\frac{\alpha}{p}}|^2_{L_2(\omega_h)}+\nu_2F^j. \tag{33}$$

Applying Lemma 4 [23] to (33), we obtain the a priori estimate (10).

4. Convergence of the LOD. For the convergence of the LOD, the following theorem holds.

Theorem 2. Suppose that the problem (1)–(3) has a unique continuous solution in $\overline{Q}_T$ solution $u(x,t)$, the continuous derivatives in $\dfrac{\partial^2u}{\partial t^2},\dfrac{\partial^4u}{\partial x_\alpha^2\partial x_\beta^2},\dfrac{\partial^3u}{\partial x_\alpha^2\partial t},\dfrac{\partial^2f}{\partial x_\alpha^2},\ 1\leq\alpha,\beta\leq p,\ \alpha\neq\beta$ exist, and the smoothness and boundedness conditions (4) are satisfied. Then, the LOD (6)–(8) converges to the solution of the original problem (1)–(3) with a rate $O(h^2+\tau)$, so that for $\tau\leq\tau_0$ the following estimate holds:

$$|[y^{j+1}-u^{j+1}]|_1\leq M\left(h^2+\tau\right), \tag{34}$$

where

$$|[z^{j+1}]|_1=\left(|\rho_p z^{j+1}|^2_{L_2(\omega_h)}+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(\|[\rho_\alpha z_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+|[z^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}\right)\right)^{1/2}.$$

Proof. Let's represent the solution of problem (9) for the error z as a sum $z_{(\alpha)}=\upsilon_{(\alpha)}+\eta_{(\alpha)},\quad z_{(\alpha)}=z^{j+\frac{\alpha}{p}}$, where $\eta_{(\alpha)}$ is defined by conditions:

$$\frac{\eta_{(\alpha)}-\eta_{(\alpha-1)}}{\tau}=\psi_\alpha,\quad x\in\omega_h+\gamma_h,\quad \alpha=1,2,...,p, \tag{35}$$

$$\eta(x,0)=0,\quad \psi_\alpha=\begin{cases}\psi_\alpha,\ x_\alpha\in\omega_h,\\ \psi_{-\alpha},\ x_\alpha=0,\\ \psi_{+\alpha},\ x_\alpha=l.\end{cases}$$

From (35), it follows that $\eta^{j+1}=\eta_{(p)}=\eta^j+\tau\left(\overset{0}{\psi}_1+\overset{0}{\psi}_2+...+\overset{0}{\psi}_p\right)=\eta^j=...=\eta^0=0,\ j=0,1,...,j_0$, так как $\eta^0=0$.

Then we have $\eta^\alpha=\tau\left(\overset{0}{\psi}_1+\overset{0}{\psi}_2+...+\overset{0}{\psi}_\alpha\right)=-\tau\left(\overset{0}{\psi}_{\alpha+1}+...+\overset{0}{\psi}_p\right)=O(\tau)$. The function $\upsilon_{(\alpha)}$ is determined by the conditions:

$$\frac{\upsilon_{(\alpha)}-\upsilon_{(\alpha-1)}}{\tau}=\Lambda_\alpha\upsilon_{(\alpha)}+\psi_\alpha,\ x\in\omega_h,\quad \alpha=1,2,...,p, \tag{36}$$

$$\upsilon_{(\alpha)}=-\eta_\alpha,\ x_\alpha\in\gamma_{h,\alpha},\ \upsilon(x,0)=0,$$

where $\widetilde{\psi}_\alpha=\overset{*}{\psi}_\alpha+\Lambda_\alpha\eta_{(\alpha)}$.

If the continuous derivatives $\dfrac{\partial^4 u}{\partial x_\alpha^2\partial x_\beta^2},\alpha\neq\beta,$ exist in the closed domain $\overline{Q}_T$ then $\Lambda_\alpha\eta_{(\alpha)}=-\tau\Lambda_\alpha\left(\overset{0}{\psi}_{\alpha+1}+...+\overset{0}{\psi}_p\right)=O(\tau)$.

We estimate the solution of problem (36) using Theorem 1.

$$|\rho_p\upsilon^{j+1}|^2_{L_2(\omega_h)}+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(|[\rho_\alpha\upsilon_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+|[\upsilon^{j'+\frac{\alpha}{p}}]|^2_{L_2(\overline{\omega}_h)}\right)\leq \tag{37}$$

$$\leq M\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(|[\widetilde{\psi}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\overline{\omega}_h)}+\sum_{i_\beta\neq i_\alpha}\left(\eta^2_{-\alpha}\left(0,x',t_{j'}\right)+\eta^2_{+\alpha}\left(l,x',t_{j'}\right)\right)\overline{H}/h\right).$$

Since $\eta^{j+1}=0,\ \eta_{(\alpha)},\quad \eta_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}}=O(\tau)$ and

$$|[z^{j+1}]|_1^2=|\rho_p z^{j+1}|^2_{L_2(\omega_h)}+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(|[\rho_\alpha z_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+|[z^{j'+\frac{\alpha}{p}}]|^2_{L_2(\overline{\omega}_h)}\right)\leq$$

$$\leq|\rho_p v^{j+1}|^2_{L_2(\omega_h)}+2\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(|[\rho_\alpha v_{\bar{x}_\alpha}^{j+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+|\rho_\alpha\eta_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+|[v^{j'+\frac{\alpha}{p}}]|^2_{L_2(\overline{\omega}_h)}+\right.$$

$$\left.+|[\eta^{j'+\frac{\alpha}{p}}]|^2_{L_2(\overline{\omega}_h)}\right)\leq 2\left(|[\upsilon^{j+1}]|_1^2+\sum_{j'=0}^{j}\tau\sum_{\alpha=1}^{p}\left(|[\rho_\alpha\eta_{\bar{x}_\alpha}^{j'+\frac{\alpha}{p}}]|^2_{L_2(\omega_h)}+|[\eta^{j'+\frac{\alpha}{p}}]|^2_{L_2(\overline{\omega}_h)}\right)\right),$$

then from estimate (37) it follows the convergence with rate (34).

Discussion and Conclusions. A numerical method for solving the first initial-boundary value problem for a multidimensional parabolic integro-differential equation has been developed and justified. To approximate the given problem, a local one-dimensional difference scheme by A.A. Samarskii with an approximation order of $O(h^2+\tau)$ has been constructed. The main idea behind the scheme construction lies in reducing the transition from one layer to another to the sequential solution of a series of one-dimensional problems along each of the coordinate directions. In this case, each of the auxiliary problems may not approximate the original problem, but in aggregate and under special norms, such approximation holds true. A variant of finding the a priori estimate of the solution for the LOD with non-homogeneous first-order boundary conditions based on the method of energy inequalities has been proposed, which is essential for implementing the investigated multidimensional problem. From this estimate, the uniqueness, continuous and uniform dependence of the solution of the local one-dimensional difference scheme on the input data, as well as the convergence of the scheme solution to the solution of the original differential problem at a rate equal to the order of approximation of the difference scheme, follow.

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