MATHEMATICAL MODELLING
МАТЕМАТИЧЕСКОЕ МОДЕЛИРОВАНИЕ

Original Theoretical Research

UDC 519.6
https://doi.org/10.23947/2587-8999-2024-8-3-23-33

A Modified Bubnov-Galerkin Method for Solving Boundary Value Problems with Linear Ordinary Differential Equations

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Abstract
Introduction. The paper considers the solution of boundary value problems on an interval for linear ordinary differential equations, in which the coefficients and the right-hand side are continuous functions. The conditions for the orthogonality of the residual equation to the coordinate functions are supplemented by a system of linearly independent boundary conditions. The number of coordinate functions m must exceed the order n of the differential equation.
Materials and Methods. To numerically solve the boundary value problem, a system of linearly independent coordinate functions is proposed on a symmetric interval $[-1,1]$, where each function has a unit Chebyshev's norm. A modified Petrov-Galerkin method is applied, incorporating linearly independent boundary conditions from the original problem into the system of linear algebraic equations. An integral quadrature formula with twelfth-order error is used to compute the scalar product of two functions.
Results. A criterion for the existence and uniqueness of a solution to the boundary value problem is obtained, provided that n linearly independent solutions of the homogeneous differential equation are known. Formulas are derived for the matrix coefficients and the coefficients of the right-hand side in the system of linear algebraic equations for the vector expansion of the solution in terms of the coordinate function system. These formulas are obtained for second- and third-order linear differential equations. The modified Bubnov-Galerkin method is formulated for differential equations of arbitrary order.
Discussion and Conclusions. he derived formulas for the generalized Bubnov-Galerkin method may be useful for solving boundary value problems involving linear ordinary differential equations. Three boundary value problems with second- and third-order differential equations are numerically solved, with the uniform norm of the residual not exceeding 10^{-11}.

Keywords: numerical methods, ordinary differential equations, boundary value problems, Galerkin method, hydrodynamics

For citation. Volosova N.K., Volosov K.A., Volosova A.K., Pastukhov D.F., Pastukhov Yu.F. A Modified Bubnov-Galerkin Method for Solving Boundary Value Problems with Linear Ordinary Differential Equations. *Computational Mathematics and Information Technologies*. 2024;8(3):23–33. https://doi.org/10.23947/2587-8999-2024-8-3-23-33

Оригинальное теоретическое исследование

Модифицированный метод Бубнова-Галеркина для решения краевых задач с линейным обыкновенным дифференциальным уравнением

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Аннотация

Введение. Рассматривается решение краевых задач на отрезке с линейными обыкновенными дифференциальными уравнениями, в которых коэффициенты и правая часть являются непрерывными функциями. Условия ортогональности невязки уравнения координатным функциям дополняются системой линейно независимых краевых условий задачи. Число координатных функций m должно быть больше порядка n дифференциального уравнения.

Материалы и методы. Для численного решения краевой задачи предложена система линейно независимых координатных функций на симметричном отрезке [–1,1] с единичной нормой Чебышева каждой функции системы. Применен модифицированный метод Петрова-Галеркина с включением линейно независимых краевых условий исходной задачи в систему линейных алгебраических уравнений. Применена интегральная квадратурная формула с двенадцатым порядком погрешности для вычисления скалярного произведения двух функций.

Результаты исследования. Получен критерий существования и единственности решения краевой задачи, при условии, что известны n линейно независимых решений однородного дифференциального уравнения. Получены формулы для матричных коэффициентов и коэффициентов правой части системы линейных алгебраических уравнений для вектора разложения решения по системе координатных функций. Формулы получены для линейных дифференциальных уравнений второго и третьего порядков. Модифицированный метод Бубнова-Галеркина сформулирован для уравнения произвольного порядка.

Обсуждение и заключение. Полученные формулы обобщенного метода Бубнова-Галеркина могут быть полезными для решения краевых задач с линейными обыкновенными дифференциальными уравнениями. Численно решены три краевых задачи с уравнениями второго и третьего порядков, равномерная норма невязки не превышает 10^{-11}.

Ключевые слова: численные методы, обыкновенные дифференциальные уравнения, краевые задачи, метод Галеркина, гидродинамика

Для цитирования. Волосова Н.К., Волосов К.А., Волосова А.К., Пастухов Д.Ф., Пастухов Ю.Ф. Модифицированный метод Бубнова-Галеркина для решения краевых задач с линейным обыкновенным дифференциальным уравнением. *Computational Mathematics and Information Technologies*. 2024;8(3):23–33. https://doi.org/10.23947/2587-8999-2024-8-3-23-33

Introduction. Boundary value problems involving ordinary differential equations can be classified by the order of the equation. For instance, in hydrodynamics problems, these equations may be of the first [1], second [2], or third order [3–4].

The most well-known methods for solving boundary value problems on an interval with ordinary differential equations are the sweep method and the shooting method [5]. In these methods, the unknown function is sought on a given grid (the so-called grid function). In this study, the solution is found in a functional form using a system of linearly independent coordinate functions that are smooth and bounded in absolute value on the symmetric interval [−1, 1]. The unknown solution function is expanded in a basis of linearly independent coordinate functions. Using the Bubnov-Galerkin method [6], where the residual of the differential or integral equation is orthogonal to the coordinate functions, the expansion coefficients of the solution are determined from the orthogonality conditions.

In [7], it was shown that in the simplest classical variational problem (a boundary value problem), the solution must be sought in the class of admissible functions defined by the boundary conditions. This idea was used by the authors in the modified Bubnov-Galerkin method, incorporating $n-1$ (where n is the order of the equation) linearly independent boundary conditions into a system of m linear algebraic equations. The number of orthogonality conditions is thus $m-n+1$ (where m is the number of coordinate functions). In this study, the modified Bubnov-Galerkin method is applied to boundary value problems involving second- and third-order equations.

Materials and Methods. Let the unknown function $u(x) \in C^n[a,b]$, which is continuously differentiable *n* times, be the solution of a boundary value problem with an ordinary differential equation of order *n* with variable coefficients $g_i(x), i = \overline{0,n}$

$$\begin{cases} L[u(x)] = f(x),\ x \in (a,b), \\ L[u(x)] \equiv \left(\sum_{i=0}^{n} g_i(x) \frac{d^i}{dx^i} \right) u(x). \end{cases} \quad (1)$$

$$\begin{cases} \sum_{i=0}^{n-1} \left(\alpha_\mu^i u^{(i)}(a) \right) = \gamma_\mu,\ \mu = \overline{1,k}, \\ \sum_{i=0}^{n-1} \left(\beta_\mu^i u^{(i)}(b) \right) = \gamma_\mu,\ \mu = \overline{k+1,n}. \end{cases} \quad (2)$$

In the boundary value problem (1)–(2), the functions $g_i(x)(i = \overline{0,n}), f(x) \in C[a,b]$ are given and continuous on the segment [*a*, *b*]. The first *k* equations in the system (2) represent the boundary conditions at point *x* = *a*, and the last *n*–*k* equations represent the boundary conditions at point *x* = *b*. For the closure of problem (1), it is necessary that the total number of boundary conditions be equal to *n*. The coefficient matrices $\alpha_\mu^i, \beta_\mu^i, i = \overline{0,n-1}, \mu = \overline{1,n}$, as well as the numbers $\gamma_\mu,\ \mu = \overline{1,n}$ are given.

Boundary conditions of the form (2) are called separated. The relationship between the numbers of boundary conditions $\alpha_\mu^i, \beta_\mu^i$ determines the existence and uniqueness of the solution of the boundary value problem (1)–(2).

Statement 1. Let *n* linearly independent particular solutions of the homogeneous equation (1) $U_j(x), j = \overline{1,n}$ be given. Then the boundary value problem (1)–(2) has a unique solution if and only if the following condition $\det A_{\mu j} \neq 0, \mu = \overline{1,n}, j = \overline{1,n}$ is satisfied:

$$A_{\mu j} = \begin{cases} \sum_{i=0}^{n-1} \alpha_\mu^i U_j^{(i)}(a), \mu = \overline{1,k} \\ \sum_{i=0}^{n-1} \beta_\mu^i U_j^{(i)}(b), \mu = \overline{k+1,n}. \end{cases}$$

Proof. Let us write the general solution of equation (1) as $u(x) = \sum_{j=1}^{n} U_j(x) D_j + \overline{u(x)}, j = \overline{1,n},$ where D_j are arbitrary integration constants, $\overline{u(x)}$ is a particular solution of the non-homogeneous equation (1), and $U_j(x)$ are linearly independent particular solutions of the homogeneous equation (1).

Substituting this solution *u*(*x*) into the boundary conditions (2):

$$\begin{aligned} &\sum_{i=0}^{n-1} \left(\alpha_\mu^i u^{(i)}(a) \right) = \sum_{i=0}^{n-1} \alpha_\mu^i \left(\sum_{j=1}^{n} U_j^{(i)}(a) D_j + \overline{u^{(i)}(a)} \right) = \gamma_\mu \Leftrightarrow, \\ &\sum_{j=1}^{n} \left(\sum_{i=0}^{n-1} \alpha_\mu^i U_j^{(i)}(a) \right) D_j = \gamma_\mu - \sum_{i=0}^{n-1} \alpha_\mu^i \overline{u^{(i)}(a)}, \mu = \overline{1,k}. \end{aligned} \quad (3)$$

Similarly, for the point *x* = *b*, we obtain:

$$\sum_{j=1}^{n} \left(\sum_{i=0}^{n-1} \beta_\mu^i U_j^{(i)}(b) \right) D_j = \gamma_\mu - \sum_{i=0}^{n-1} \beta_\mu^i \overline{u^{(i)}(b)}, \mu = \overline{k+1,n}. \quad (4)$$

The resulting non-homogeneous system of *n* linear algebraic equations (3)–(4) with respect to the *n* unknowns D_j, *j* = 1, *n* as a unique solution if and only if the determinant of the matrix $\det A_{\mu j} \neq 0, \mu = \overline{1,n}, j = \overline{1,n}$, where

$$A_{\mu j} = \begin{cases} \sum_{i=0}^{n-1} \alpha_\mu^i U_j^{(i)}(a), \mu = \overline{1,k} \\ \sum_{i=0}^{n-1} \beta_\mu^i U_j^{(i)}(b), \mu = \overline{k+1,n}. \end{cases} \quad (5)$$

Statement 1 is proven.

Let us now consider a simple case of problem (1) involving a second-order ordinary differential equation (ODE) with Dirichlet boundary conditions:

$$\begin{cases} L[u(x)] = f(x),\ x \in (a,b) \\ L[u(x)] \equiv \left[g_2(x)\dfrac{d^2}{dx^2} + g_1(x)\dfrac{d}{dx} + g_0(x) \right] u(x) \\ u(a) = u_a, u(b) = u_b. \end{cases} \tag{6}$$

Let us generalize the Bubnov-Galerkin method, as proposed in work [6] for solving Fredholm integral equations of the second kind, to the solution of the Dirichlet problem with the second-order ODE (6).

We begin by selecting a system of basis (coordinate) functions $\varphi_i(x)$:

$$\{\varphi_i(x)\}_{i=0}^{m} = \left\{ \left(\frac{2x-a-b}{b-a} \right)^i, x \in [a,b], i = \overline{0,m} \right\}. \tag{7}$$

Statement 2. The coordinate functions of the system (7) $\varphi_i(x) \in C^{\infty}[a,b]$ are bounded in modulus, differentiable any number of times, and linearly independent.

Proof will be conducted by contradiction. We use a linear mapping $z = \dfrac{2x-a-b}{b-a} \in [-1,1], x \in [a,b]$, which bijectively maps the interval $x \in [a,b]$ onto a symmetric interval $z \in [-1,1]$. Such a straightforward method is employed by the authors of the textbook [5] in the task of constructing integral quadrature formulas. Assume that the system of coordinate functions is linearly dependent, and taking into account the variable z it takes the form $\left\{ \varphi_i(z) = z^i, z \in [-1,1], i = \overline{0,m} \right\}$. If the system of functions is linearly dependent, then there exists a non-trivial solution $(\alpha_0, \alpha_1, \dots \alpha_m)$ to the equation $\alpha_0 + \alpha_1 z + \alpha_2 z^2 + \dots + \alpha_m z^m \equiv 0 \forall z \in [-1,1]$.

The last equation has no more than m real solutions, whereas a solution is required for all points of the interval $z \in [-1,1]$. This contradiction proves the linear independence of the functions in system (7). The functions in (7) are infinitely continuously differentiable with respect to the variable x as they are polynomials of finite degree, and they are also bounded since $\|\varphi_i\|_C = \max\limits_{z \in [-1,1]} |z^i| = 1$. **Statement 2 is proven.**

We will apply the Bubnov-Galerkin method using the system of linearly independent coordinate functions (7) to solve the Dirichlet boundary value problem (6). The symmetric interval $z \in [-1,1]$ in our problem results in a consistent order of error at the nodes symmetrically located with respect to the midpoint of the interval $c = (a+b)/2$ and generally reduces the norm of the error.

Let us express the solution as a series expansion in terms of the linearly independent system of coordinate functions:

$$u(x) = u(c) + \sum_{j=1}^{m} \varphi_j(x) C_j = u(c) + \sum_{j=1}^{m} \left(\frac{2(x-c)}{b-a} \right)^j C_j. \tag{8}$$

In equation (8), the coefficients C_j are unknown and need to be determined.

From equation (8), we derive the identity $u(c) = u(c)$, which resembles the expansion of an unknown function in a Taylor series centered at some point $x = c = (a+b)/2$, although we do not know either the function itself or its derivatives. Substituting equation (8) into equation (6), we obtain the residual (discrepancy) of equation (6):

$$R(u((x)) = L[u(x)] - f(x) = L\left(u(c) + \sum_{j=1}^{m} \varphi_j(x) C_j \right) - f(x) = L\big(u(c)\big) + \sum_{j=1}^{m} L\varphi_j(x) C_j - f(x).$$

The Bubnov-Galerkin method is orthogonal, so we require the residual to be orthogonal to the maximum number of coordinate functions, $\left\{1, z, z^2, \dots, z^{m-2}\right\}$. Specifically, we impose orthogonality with respect to m–1 functions that contribute the most to the residual of equation (6):

$$\left\langle R\big(u(x)\big), \varphi_i(x) \right\rangle = 0, \overline{i = 0, m-2} \Leftrightarrow \sum_{j=1}^{n} \left\langle L\varphi_j(x), \varphi_i(x) \right\rangle C_j = \left\langle f(x) - L\big(u(c)\big), \varphi_i(x) \right\rangle, i = \overline{0, m-2}. \tag{9}$$

In equation (9), we introduce the notation:

$$\left\langle f, g \right\rangle = \int_a^b f(x) g(x) dx,\ L\big(u(c)\big) = g_0(x) u(c) = g_0(x) u_c.$$

Unlike the method described in [6, p. 140], the last condition, numbered m, for the system of linear algebraic equations (SLAE) with respect to m unknowns $C_j,\ j = \overline{1,m}$, will be derived from the boundary conditions

$$\frac{u_b - u_a}{2} = C_1 + C_3 + \dots + \begin{cases} C_{m-1}, m = 2l \\ C_m, m = 2l+1. \end{cases} \tag{10}$$

Let us demonstrate the validity of equation (10). At the endpoints of the interval, specifically at the points $x = a, x = b$, we can use the expansion given in equation (8) to obtain:

$$u(a) \equiv u_a = u(c) + \sum_{j=1}^{m} \left(\frac{(2a-a-b)}{b-a} \right)^j C_j = u_c + \sum_{j=1}^{m} (-1)^j C_j, u(b) \equiv u_b = u(c) + \sum_{j=1}^{m} \left(\frac{(2b-a-b)}{b-a} \right)^j C_j = u_c + \sum_{j=1}^{m} C_j .$$

By summing the two most recent equations and expressing $u(c) = u_c$, we obtain

$$u_c = \left(\frac{u_a + u_b}{2} \right) - C_2 - C_4 - \ldots - \begin{cases} C_m, m = 2l \\ C_{m-1}, m = 2l+1. \end{cases} \quad (11)$$

Similarly, by subtracting the first equation u_b from the second equation u_a and expressing $\frac{u_b - u_a}{2}$, we obtain equation (10). Next, we substitute the value $u(c)$ obtained from equation (11) into the right-hand side of equation (9). Then, we move all terms involving C_j to the left-hand side of equation (9) to obtain the system of linear algebraic equations (SLAE) for the coefficients C_j:

$$\sum_{j=1}^{m} a_{i,j} C_j = \overline{f_i}, \ i = \overline{0, m-1}. \quad (12)$$

The elements of the matrix $a_{i,j}, i = \overline{0, m-1}, j = \overline{1, m}$ and the coefficients on the right-hand side $\overline{f_i}$ in the system of equations (12) are defined as follows:

$$a_{i,j} = \begin{cases} \left\langle L\varphi_j, \varphi_i \right\rangle, \text{если}\, j \equiv 1 (\text{mod}\, 2), i = \overline{0, m=2} \\ \left\langle L(\varphi_j - 1), \varphi_i \right\rangle, \text{если}\, j \equiv 0 (\text{mod}\, 2), i = \overline{0, m-2} \\ 1, \text{если}\ i = m-1,\ j \equiv 1 (\text{mod}\, 2) \\ 0, \text{если}\ i = m-1,\ j \equiv 1 (\text{mod}\, 2) \end{cases},$$

$$\overline{f_i} = \begin{cases} \left\langle f(x) - L\left(\frac{u_a + u_b}{2} \right), \varphi_i(x) \right\rangle, \text{если}\ i = \overline{0, m-2} \\ \frac{u_b - u_a}{2}, \text{если}\ i = m-1 \end{cases},$$

$$L\left(\frac{u_a + u_b}{2} \right) = \left(\frac{u_a + u_b}{2} \right) g_0(x).$$

Remark 1. It is not possible to use both Dirichlet boundary conditions $u(a)$, $u(b)$ directly in the system of linear algebraic equations (12) because these conditions are linearly dependent.

Proof. Let us substitute the value of $u(c) = u_c$ from equation (11) into the expressions for $u(a)$, $u(b)$:

$$u(c) = \left(\frac{u_a + u_b}{2} \right) - C_2 - C_4 - \ldots - \begin{cases} C_m, m = 2l \\ C_{m-1}, m = 2k+1 \end{cases}, u_a = u_c + \sum_{j=1}^{m} (-1)^j C_j = \left(\frac{u_a + u_b}{2} \right) - \left(C_1 + C_3 + \ldots + \begin{cases} C_{m-1}, m = 2l \\ C_m, m = 2l+1 \end{cases} \right).$$

The last expression is equivalent to (10).

$$u_b = u_c + \sum_{j=1}^{m} C_j = \left(\frac{u_a + u_b}{2} \right) + C_1 + C_3 + \ldots + \begin{cases} C_{m-1}, m = 2l \\ C_m, m = 2l+1. \end{cases}$$

The last equation is equivalent to equation (10), which proves the linear dependence of the boundary conditions.

Remark 2. In equations (12), for the matrix coefficients a_{ij} in even columns, the differential operator L acts on the non-positive function $\varphi_j(x)$–1, and in odd columns on the alternating coordinate function $\varphi_j(x)$. If the determinant of the matrix in the SLAE (12) is non-zero, then the numerical solution of (12) is unique. Let us now write the differentiation formulas for the linear operator L as defined in equation (6), applied to the coordinate functions from equation (8):

$$\begin{cases} L\varphi_0 = g_0(x), \text{если}\, j = 0, \\ L\varphi_1 = \frac{2g_1(x)}{(b-a)} + g_0(x)\left(\frac{2x-a-b}{b-a} \right), \text{если}\, j = 1, \\ L\varphi_j = 4j(j-1)g_2(x)\frac{(2x-a-b)^{j-2}}{(b-a)^j} + 2jg_1(x)\frac{(2x-a-b)^{j-1}}{(b-a)^j} + g_0(x)\left(\frac{2x-a-b}{b-a} \right)^j, \text{если}\, j \geq 2. \end{cases} \quad (13)$$

Considering (11), the numerical solution of the Dirichlet problem (6) can be reduced to expression (14) by converting formula (8):

$$u(x)=\left(\frac{u_a+u_b}{2}\right)+\sum_{j=1}^{m}\left[\left(\frac{(2x-a-b}{b-a}\right)^j+\left(\frac{-1+(-1)^{j+1}}{2}\right)\right]C_j. \quad (14)$$

It follows from (12) that the vector C included in formula (14) has the form $C = A^{-1}\overline{f}$.

Let us estimate in absolute value $u(x)$ based on the given formula $C = A^{-1}\overline{f}$

$$|u(x)|\le\frac{|u_a|+|u_b|}{2}+2\sum_{j=1}^{m}|C_j|\le\frac{|u_a|+|u_b|}{2}+2m\max_{\overline{j=1,m}}C_j=\frac{|u_a|+|u_b|}{2}+2m\|C\|_C\le\frac{|u_a|+|u_b|}{2}+2m\left\|A^{-1}\right\|_C\|f\|_C\Rightarrow$$

$$\|u\|_C\le\frac{|u_a|+|u_b|}{2}+2m\left\|A^{-1}\right\|_C\left\|\overline{f}\right\|_C.$$

It is known that the norm $\|B\|_C$ of an arbitrary square matrix $B(m\times m)$ is determined by the formula $\|B\|_C=\max_{\overline{i=1,m}}\sum_{j=1}^{m}|b_{i,j}|$.

In [9], a composite quadrature integral formula with a uniform step and with the 12th order of error $O(h^{12})$ is obtained, which is used by the program to calculate all matrix elements a_{ij}, as well as the coefficients of the right side: $\overline{f_i}$ of SLAE (12) through the scalar product of two functions:

$$\langle y_1,y_2\rangle=\int_a^b y_1(x)y_2(x)dx=5h\sum_{i=0}^{n_1}y_1(x_i)y_2(x_i)C_i+O(h^{12}),\ n_1=10p,\ h=\frac{b-a}{n_1},\ p\in N, \quad (15)$$

where the weight coefficients of the integral quadrature formula (15) are determined by the value of the remainder modulo 10 of the node number of the uniform grid i:

$$C_i=\begin{cases}\frac{16067}{299376},\text{если } i=0 \text{ или } i=n_1,\\ \frac{16067}{149688},\text{если } i\equiv0\ (\mathrm{mod}10)\text{ и }(0<i<n_1),\\ \frac{26575}{74844},\text{если } i\equiv1\ (\mathrm{mod}10)\text{ или } i\equiv9\ (\mathrm{mod}10),\\ \frac{-16175}{99792},\text{если } i\equiv2\ (\mathrm{mod}10)\text{ или } i\equiv8\ (\mathrm{mod}10),\\ \frac{5675}{6237},\text{если } i\equiv3\ (\mathrm{mod}10)\text{ или } i\equiv7\ (\mathrm{mod}10),\\ \frac{-4825}{5544},\text{если } i\equiv4\ (\mathrm{mod}10)\text{ или } i\equiv6\ (\mathrm{mod}10),\\ \frac{17807}{12474},\text{если } i\equiv5\ (\mathrm{mod}10).\end{cases}$$

Here are examples of numerical solution of boundary value problems by the algorithm (12)–(15).

Example 1 [10]. Solve the Dirichlet boundary value problem (16)

$$y''-y=2x,y(0)=0,y(1)=-1,x\in[0,1]. \quad (16)$$

The exact solution $y(x)=sh(x)/sh(1)-2x$.

A program in the Fortran language, where functions and variables are set with double precision according to the algorithm (12)–(15), gives the Chebyshev vector norm of the difference between the exact and approximate solution $\|y-u\|_C=4{,}218847493575595\mathrm{E}-015$, if the number of coordinate functions $m=11$, the number of intervals for calculating the scalar product of functions by formula (15) on a uniform grid is $n_1=50$, $\|y-u\|_C=\max_{i=0,n_1}|y(x_i)-u(x_i)|,x_i=a+hi,h=\frac{b-a}{n_1}$.

The inverse matrix A^{-1} n the system of linear algebraic equations (12) is calculated by the msimsl linear algebra library to find the vector of expansion coefficients $C_j,j=\overline{1,m}$.

Example 2 [9]. Solve the Dirichlet problem for the Poisson equation on a rectangle

$$\begin{cases}u_{xx}+u_{yy}=e^y\sin x,\ 0<x<\pi,0<y<\pi\\ u|_{x=0}=u|_{x=\pi}=u|_{y=0}=u|_{y=\pi}=0.\end{cases}$$

We are looking for a solution to the problem in the form $u(x)=\sin(x)f(y)$. This choice of solution automatically fulfills two boundary conditions $u|_{y=0}=u|_{y=\pi}=0$. Substituting the solution $u(x)$ into the Poisson equation $\sin(x)\left(f''(y)-f(y)\right)=e^{y}\sin(x), \forall x\in(0,\pi)$, we obtain the Dirichlet boundary value problem for $f(y)$:

$$\begin{cases} f''(y)-f(y)=e^{y} \\ f(0)=f(\pi)=0. \end{cases} \quad (17)$$

The last Dirichlet boundary condition $f(0)=f(\pi)=0$ in (17) fulfills the boundary conditions of the original problem $u|_{y=0}=u|_{y=\pi}=0$.

The general solution of the homogeneous equation (17) $f''(y)-f(y)=0$ can be written as $f_{o,o}(y)=A\,\mathrm{ch}(y)+B\,\mathrm{sh}(y)$, and the partial solution of the inhomogeneous equation is sought in the form

$$f_{ч}(y)=Cye^{y},\ f_{ч}''(y)=Ce^{y}\left(y+2\right),\ f_{ч}''-f_{ч}=Ce^{y}\left(y+2\right)-Cye^{y}=e^{y}\Leftrightarrow 2C=1, C=\frac{1}{2}.$$

Let's write down the general solution of the inhomogeneous equation (17) as

$$f_{o,н}(y)=A\,\mathrm{ch}(y)+B\,\mathrm{sh}(y)+\frac{ye^{y}}{2},\ f_{o,н}(0)=0\Rightarrow A=0,\ f_{o,н}(\pi)=0\Rightarrow B=\frac{-\pi e^{\pi}}{2\,\mathrm{sh}(\pi)},$$

$f(y)=\dfrac{ye^{y}\,\mathrm{sh}(\pi)-\pi e^{\pi}\,\mathrm{sh}(y)}{2\,\mathrm{sh}(\pi)},\ u(x,y)=\left(\dfrac{ye^{y}\,\mathrm{sh}(\pi)-\pi e^{\pi}\,\mathrm{sh}(y)}{2\,\mathrm{sh}(\pi)}\right)\sin(x)$ the exact solution of the problem from Example 2.

Solving numerically the boundary value problem (17) using the algorithm (12)–(15), we obtain the Chebyshev norm for the difference between the numerical and approximate solutions with the number of coordinate functions $m=11$, the number of intervals for calculating the scalar product of functions on a uniform grid $n_1=100$, $\|f-f_{num}\|_C=8.079448221565144\mathrm{E}-011$.

Let's estimate the uniform rate of computational error in Example 2 using the algorithm (12)–(15) $\|u-u_{num}\|_C\le\|f-f_{num}\|_C\,\|\sin(x)\|_C=\|f-f_{num}\|_C\approx 8\cdot10^{-11}$.

In hydrodynamics [3, 4], boundary value problems with a third-order differential equation are encountered. Consider example 3.

Example 3.

$$\begin{cases} u'''(x)+u'(x)=-2\sin(x), x\in(0,\pi), \\ u(0)=0, u'(0)=0, y(\pi)=0. \end{cases} \quad (18)$$

Let 's solve the homogeneous equation $u'''(x)+u'(x)=0$. Its characteristic equation and eigenvalues are equal $\lambda^3+\lambda=0\Leftrightarrow\lambda_1=0,\lambda_{2,3}=\pm i=\pm\sqrt{-1}$, which correspond to 3 partial linearly independent solutions

$$\left\{U_1(x)=1, U_2(x)=\sin(x), U_3(x)=\cos(x)\right\}, \left\{U_1'(x)=0, U_2'(x)=\cos(x), U_3'(x)=-\sin(x)\right\},$$

$$\left\{U_1''(x)=0, U_2''(x)=-\sin(x), U_3''(x)=-\cos(x)\right\}.$$

Let's check the existence and uniqueness of the solution of the boundary value problem (18). Write down the elements of the matrix according to the formula (5):

$$\alpha_1^0=1;\alpha_1^1=0;\alpha_1^2=0;\alpha_2^0=0;\alpha_2^1=1;\alpha_2^2=0;\beta_3^0=1;\beta_3^1=0;\beta_3^2=0,$$

$$A_{\mu j}=\begin{cases} \sum_{i=0}^{n-1}\alpha_{\mu}^{i}U_j^{(i)}(a), \mu=\overline{1,k} \\ \sum_{i=0}^{n-1}\beta_{\mu}^{i}U_j^{(i)}(b), \mu=\overline{k+1,n},\ k=2, n=3. \end{cases}$$

$$A_{11}=1\cdot1+0\cdot0+0\cdot0=1, A_{21}=0\cdot1+1\cdot0+0\cdot0=0, A_{31}=1\cdot1+0\cdot0+0\cdot0=1,$$

$$A_{12}=1\cdot\sin(0)+0\cdot\cos(0)+0\cdot(-\sin(0))=0, A_{22}=0\cdot\sin(0)+1\cdot\cos(0)+0\cdot(-\sin(0))=1,$$

$$A_{32}=1\cdot\sin(\pi)+0\cdot\cos(\pi)+0\cdot(-\sin(\pi))=0, A_{13}=1\cdot\cos(0)+0\cdot(-\sin(0))+0\cdot(-\cos(0))=1,$$

$$A_{23}=0\cdot\cos(0)+1\cdot(-\sin(0))+0\cdot(-\cos(0))=0, A_{33}=1\cdot\cos(\pi)+0\cdot(-\sin(\pi))+0\cdot(-\cos(\pi))=-1.$$

Since $\begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{vmatrix} = -2 \neq 0$, the boundary value problem (18) has a unique solution.

By direct verification, we will make sure that the exact solution of the boundary value problem (18) is the function

$$u(x) = x\sin(x), u'(x) = \sin(x) + x\cos(x), u''(x) = 2\cos(x) - x\sin(x),$$

$$u'''(x) = -3\sin(x) - x\cos(x), u'''(x) + u'(x) = -3\sin(x) - x\cos(x) + \sin(x) + x\cos(x) = -2\sin(x), u(0) = u(\pi) = u'(0) = 0.$$

Statement 1 for boundary value problem (18) is fulfilled, therefore, the solution of the problem is unique and coincides with $u(x) = x\sin(x)$. There are no other solutions.

Let's calculate the first derivative $u(x)$ by formula (8) and equate it to zero at the point $x = a$.

$$u'(x) = \sum_{j=1}^{m} \varphi'_j(x) C_j = \sum_{j=1}^{m} \frac{2j}{(b-a)} \left(\frac{(2x-a-b)}{b-a} \right)^{j-1}_{x=a} C_j = 0 \Leftrightarrow C_1 - 2C_2 + 3C_3 + \ldots + m(-1)^{m-1} C_m = 0. \quad (19)$$

For a boundary value problem with a third-order differential equation (18), we obtain a system of equations

$$\sum_{j=1}^{m} a_{i,j} C_j = \overline{f_i},\ i = \overline{0, m-1}. \quad (20)$$

$$a_{i,j} = \begin{cases} \left\langle L\varphi_j, \varphi_i \right\rangle, \text{if } j \equiv 1 (\operatorname{mod} 2), i = \overline{0, m-3} \\ \left\langle L(\varphi_j - 1), \varphi_i \right\rangle, \text{if } j \equiv 0 (\operatorname{mod} 2), i = \overline{0, m-3} \\ 1, \text{ if } i = m-2, j \equiv 1 (\operatorname{mod} 2) \\ 0, \text{ if } i = m-2, j \equiv 0 (\operatorname{mod} 2) \\ j(-1)^{j-1}, \text{ if } i = m-1 \end{cases},$$

$$\overline{f_i} = \begin{cases} \left\langle f(x) - L\left(\frac{u_a + u_b}{2} \right), \varphi_i(x) \right\rangle, \text{ if } i = \overline{0, m-3} \\ \frac{u_b - u_a}{2}, \text{ if } i = m-2 \\ 0, \text{ if } i = m-1 \end{cases},$$

$$\begin{cases} L\varphi_0 = g_0(x), \text{if } j = 0, \\ L\varphi_1 = \frac{2g_1(x)}{(b-a)} + g_0(x) \left(\frac{2x-a-b}{b-a} \right), \text{if } j = 1, \\ L\varphi_2 = 8g_2(x) \frac{1}{(b-a)^2} + 4g_1(x) \frac{(2x-a-b)}{(b-a)^2} + g_0(x) \left(\frac{2x-a-b}{b-a} \right)^2, \text{if } j = 2, \\ L\varphi_j = 8j(j-1)(j-2) g_2(x) \frac{(2x-a-b)^{j-3}}{(b-a)^j} + \\ +4j(j-1) g_2(x) \frac{(2x-a-b)^{j-2}}{(b-a)^j} + 2j g_1(x) \frac{(2x-a-b)^{j-1}}{(b-a)^j} + g_0(x) \frac{(2x-a-b)^j}{(b-a)^j}, \text{if } j \geq 3. \end{cases} \quad (21)$$

$$u(x) = \left(\frac{u_a + u_b}{2} \right) + \sum_{j=1}^{m} \left[\left(\frac{(2x-a-b)}{b-a} \right)^j + \left(\frac{-1 + (-1)^{j+1}}{2} \right) \right] C_j. \quad (22)$$

The inverse matrix A^{-1} is calculated by the msimsl linear algebra library to find the vector of expansion coefficients $C_j, j = \overline{1, m}$, using the coefficients of the system of linear algebraic equations (20). A program using formulas (14), (20), (21), (22) gives a numerical u_i^{num} and exact $u_i^{exact} = x_i \sin(x_i)$ solution to problem (18) on a uniform grid $x_i = a + h \cdot i, i = \overline{0, n_1}, h = \frac{b-a}{n_1}, n_1 = 50, a = 0, b = \pi$. The number of coordinate functions is $m = 15$. The numerical and exact solution of this problem is presented in Table 1.

Table 1

Problem solution (18)

x_i	u_i^{num}	u_i^{exact}	$u_i^{num} - u_i^{exact}$
0.000000000E+000	0.0000000000E+000	0.00000000000E+000	0.00000000E+000
0.12566370614359	1.5749838632E–002	1.5749838632E–002	3.36702887793E–013
0.25132741228718	6.2502585803E-002	6.2502585803E–002	–7.5051076464E–014
0.37699111843077	0.1387796868382	0.1387796868384	–2.2543078515E–013
0.50265482457436	0.2421558085434	0.2421558085436	–2.5310309403E–013
0.62831853071795	0.3693163660978	0.3693163660980	–2.3742119381E–013
0.75398223686155	0.5161363581649	0.5161363581652	–2.1926904736E–013
0.87964594300514	0.6777788480392	0.6777788480394	–2.0117241206E–013
1.00530964914873	0.8488110105527	0.8488110105529	–1.7474910407E–013
1.13097335529233	1.0233352874866	1.0233352874867	–1.4477308241E–013
1.25663706143592	1.1951328658964	1.1951328658966	–1.3122836151E–013
1.38230076757951	1.3578164206656	1.3578164206658	–1.4432899320E–013
1.50796447372310	1.5049888502957	1.5049888502959	–1.6875389974E–013
1.63362817986669	1.6304045878204	1.6304045878205	–1.7497114868E–013
1.75929188601028	1.72812998993818	1.72812998993833	–1.5254464358E–013
1.88495559215388	1.79269929884481	1.79269929884493	–1.2145839889E–013
2.01061929829747	1.81926273330968	1.81926273330979	–1.1013412404E–013
2.13628300444106	1.80372339742481	1.80372339742493	–1,2212453270E–013
2.26194671058465	1.74285989495849	1.74285989495861	–1.2412293415E–013
2.38761041672824	1.63443180085643	1.63443180085651	–8.038014698286E–014
2.51327412287183	1.47726546439236	1.47726546439237	–5.1070259132E–015
2.63893782901543	1.27131799485423	1.27131799485419	4.50750547997E–014
2.76460153515902	1.01771770348181	1.01771770348179	2.17603712826E–014
2.89026524130261	0.71877973673595	0.71877973673604	–9.7144514654E–014
3.01592894744620	0.37799612718318	0.37799612718362	–4.3676173788E–013
3.07876080051800	0.193316990170226	0.193316990171009	–7.8290152139E–013
3.14159265358979	3.8472143247E-016	–1.0104259667E-015	1.39514739920E–015

The first column of Table 1 shows the value of a node x_i of a uniform grid, the second column contains a numerical solution u_i^{num}, and the third column contains the exact solution u_i^{exact} in nodes x_i. The last column contains their difference $u_i^{num} - u_i^{exact}$.

In Example 3, the program gives the error rate $\left\| u_i^{num} - u_i^{exact} \right\|_C = \max_{i=0,n_1} \left| u_i^{num} - u_i^{exact} \right| \approx 7.829\text{E} - 013.$

Results. The authors have developed the following algorithm for the modified Bubnov-Galerkin method:

– in the boundary value problem with an ordinary differential equation of order *n* it is necessary to select a system of *m*+1 coordinate functions $\{1, z, z^2, ..., z^m, m > n\}$;

– from the n boundary conditions, choose a system of linearly independent conditions (in the case of specified function values u_a, u_b there are *n*–1), independent conditions) and include the independent boundary conditions in the system of linear algebraic equations (SLAE);

– require that the first *m*–(*n*–1) = *m*–*n*+1 coordinate functions should be orthogonal to the residual of the differential equation. Then, the non-homogeneous system of linear algebraic equations will have *m*–*n*+1+*n*–1 = *m* rows and m unknowns $C_j, j = \overline{1,m}.$

Discussion and Conclusions. The main results obtained by the authors are as follows:

1. A system of coordinate functions is proposed that is infinitely differentiable, bounded, and linearly independent on the interval [−1,1], designed for solving boundary value problems with a linear differential equation of order *n*.

2. For the first time, a modified Bubnov-Galerkin method is introduced, in which the system of linear algebraic equations (12), (20) includes *n*–1 boundary conditions of the problem.

3. A criterion (5) for the existence and uniqueness of the solution to the boundary value problem with separated boundary conditions is obtained for the case where n linearly independent solutions of the linear homogeneous differential equation are known (Statement 1).

4. The modified Bubnov-Galerkin algorithm is proposed for boundary value problems with second- and third-order equations (12)–(15) and (20)–(22).

5. Three examples have been numerically solved using the modified algorithm, achieving a uniform error norm of no more than 10^{-11}.

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Contributions of the authors: **all authors have made an equivalent contribution to the preparation of the publication.**

Conflict of Interest Statement: the authors declare no conflict of interest.

All authors have read and approved the final manuscript.

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Заявленный вклад авторов: **все авторы сделали эквивалентный вклад в подготовку публикации.**

Конфликт интересов: **авторы заявляют об отсутствии конфликта интересов.**

Все авторы прочитали и одобрили окончательный вариант рукописи.

Received / Поступила в редакцию 30.07.2024
Revised / Поступила после рецензирования 19.08.2024
Accepted / Принята к публикации 26.08.2024