COMPUTATIONAL MATHEMATICS
ВЫЧИСЛИТЕЛЬНАЯ МАТЕМАТИКА

Original Theoretical Research

UDC 519.6
https://doi.org/10.23947/2587-8999-2024-8-3-43-59

Construction of Second-Order Finite Difference Schemes for Diffusion-Convection Problems of Multifractional Suspensions in Coastal Marine Systems

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Abstract

Introduction. This paper addresses an initial-boundary value problem for the transport of multifractional suspensions applied to coastal marine systems. This problem describes the processes of transport, deposition of suspension particles, and the transitions between its various fractions. To obtain monotonic finite difference schemes for diffusion-convection problems of suspensions, it is advisable to use schemes that satisfy the maximum principle. When constructing a finite difference scheme that adheres to the maximum principle, it is desirable to achieve second-order spatial accuracy for both interior and boundary points of the domain under study.

Materials and Methods. This problem presents certain difficulties when considering the boundaries of the geometric domain, where boundary conditions of the second and third kinds are applied. In these cases, to maintain second-order approximation accuracy, an "extended" grid is introduced (a grid supplemented with fictitious nodes). The guideline is the approximation of the given boundary conditions using the central difference formula, with the exclusion of the concentration function at the fictitious node from the resulting expressions.

Results. Second-order accurate finite difference schemes for the diffusion-convection problem of multifractional suspensions in coastal marine systems are constructed.

Discussion and Conclusion. The proposed schemes are not absolutely stable, and a detailed analysis of stability and convergence, particularly concerning the grid step ratio, remains an important problem that the author plans to address in the future.

Keywords: coastal marine systems, multifractional suspension, diffusion-convection problem, difference scheme, approximation error

Funding information. The study was supported by the Russian Science Foundation grant no. 23–21–00509, https://rscf.ru/project/23-21-00509

For citation. Sidoryakina V.V. Construction of Second-Order Finite Difference Schemes for Diffusion-Convection Problems of Multifract Suspensions in Coastal Marine Systems. *Computational Mathematics and Information Technologies*. 2024;8(3):43–59. https://doi.org/10.23947/2587-8999-2024-8-3-43-59

Оригинальное теоретическое исследование

Построение разностных схем второго порядка точности для задач диффузии-конвекции мультифракционных взвесей в прибрежных морских системах

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Аннотация

Введение. Рассматривается начально-краевая задача транспорта мультифракционных взвесей применительно к прибрежным морским системам. Данная задача описывает процессы переноса и осаждения частиц взвеси, а также взаимный переход между её различными фракциями. С целью получения монотонных разностных схем для задач диффу-

зии-конвекции взвесей целесообразно использовать разностные схемы, удовлетворяющие принципу максимума. При построении разностной схемы, для которой будет выполнен принцип максимума, желательно получить второй порядок аппроксимации по пространственной переменной как для внутренних, так и для граничных точек исследуемой области.
Материалы и методы. Данная задача вызывает определенные трудности при рассмотрении границ геометрической области, для которых выполнены граничные условия второго и третьего рода. В этих случаях, чтобы сохранить второй порядок погрешности аппроксимации, вводится «расширенная» сетка (сетка, дополненная фиктивными узлами). Ориентиром служит аппроксимация указанных граничных условий по формуле центральных разностей и исключение из полученных выражений функций концентрации взвеси в фиктивном узле.
Результаты исследования. Построены разностные схемы второго порядка точности для задачи диффузии-конвекции мультифракционных взвесей в прибрежных морских системах.
Обсуждение и заключение. Предложенные схемы не являются абсолютно стабильными и подробный анализ устойчивости и сходимости, связанный с отношением шагов сетки, является важной проблемой, которую автор планирует решать в будущем.

Ключевые слова: прибрежные морские системы, мультифракционная взвесь, задача диффузии-конвекции, разностная схема, погрешность аппроксимации

Финансирование. Исследование выполнено за счет гранта Российского научного фонда № 23–21–00509, https://rscf.ru/project/23-21-00509

Для цитирования. Сидорякина В.В. Построение разностных схем второго порядка точности для задач диффузии-конвекции мультифракционных взвесей в прибрежных морских системах. *Computational Mathematics and Information Technologies.* 2024;8(3):43–59. https://doi.org/10.23947/2587-8999-2024-8-3-43-59

Introduction. Suspended matter (suspension) is a natural component of marine systems. Changes in the quantitative and qualitative composition of the suspension can shape the landscape, negatively affect ecological communities, and shorten the lifespan of infrastructure. To address these issues, a clear understanding of the transport processes of suspended matter, accounting for spatial and temporal variations, is necessary. Typically, mathematical and numerical modelling methods are employed for these purposes [1–4].

In this article, we present a mathematical model of suspension transport based on a three-dimensional diffusion-convection equation. The model considers the multifractional composition of the suspension, water flow velocity, hydraulic particle size, complex bottom geometry, wind stress, bed friction, and other factors [5–8]. Special attention is paid to the approximation of the proposed model at both internal and boundary points of the computational domain. The proposed methods enable the construction of a finite difference scheme that approximates the model with second-order accuracy in relation to the spatial grid steps, taking into account boundary conditions of the second and third kinds.

Materials and Methods

1. Formulation of the Diffusion-Convection Problem for Multifractional Suspensions. In a rectangular Cartesian coordinate system, we consider the three-dimensional diffusion-convection equation using a skew-symmetric form of the convective transport operator [5–7]:

$$\frac{\partial c_r}{\partial t}+C_0c_r=Dc_r+F_r,\; r=1,2,3,\; (x,y,z)\in\overline{G},\; \overline{G}=\{0\le x\le L_x, 0\le y\le L_y, 0\le z\le L_z\};$$

$$C_0c_r\equiv\frac{1}{2}\left[u\frac{\partial c_r}{\partial x}+v\frac{\partial c_r}{\partial y}+\frac{\partial c_r}{\partial z}+w_r'\frac{\partial(uc_r)}{\partial x}+\frac{\partial(vc_r)}{\partial y}+\frac{\partial(w_r'c_r)}{\partial z}\right],$$

$$Dc_r=\frac{\partial}{\partial x}\left(\mu_{h,r}\frac{\partial c_r}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu_{h,r}\frac{\partial c_r}{\partial y}\right)+\frac{\partial}{\partial z}\left(\mu_{v,r}\frac{\partial c_r}{\partial z}\right),\tag{1}$$

$$F_1=(\alpha_2c_2(x,y,z,t)-\beta_1c_1)+\gamma_1c_1,$$

$$F_2=(\beta_1c_1(x,y,z,t)-\alpha_2c_2)+(\alpha_3c_3(x,y,z,t)-\beta_2c_2)+\gamma_2c_2,$$

$$F_3=(\beta_2c_2(x,y,z,t)-\alpha_3c_3)+\gamma_3c_3,$$

where c_r, $c_r=c_r(x,y,z,t)$ is the concentration of particles at time t, $t\in[0;T]$; u, v, w are the components of the velocity vector of the water medium $\vec{U}$; w_r', $w_r'=w+w_{g,r}$ are the hydraulic sizes of the particles; $\mu_{h,r},\mu_{v,r}$ are the horizontal and vertical diffusion coefficients of the particles, respectively; F_r is the source term; $\alpha_r\beta_r$ are the coefficients describing the intensity of conversion of particles from one fraction to another, and $\alpha_r\ge 0$, $\beta_r\ge 0$; γ_r is the external source power of particles. Here, the subscript r indicates that the particle belongs to fraction number r.

The equation (1) is supplemented by the initial conditions:

$$c_r(x,y,z,0)=c_{r,0}(x,y,z),\ (x,y,z)\in\overline{G}; \tag{2}$$

and the boundary conditions:

− on the lateral faces of the parallelepiped G:

$$c_r=c_r',\ \text{if}\ u_{\vec{n}}<0; \tag{3}$$

$$\frac{\partial c_r}{\partial \vec{n}}=0,\ \text{if}\ u_{\vec{n}}\geq 0; \tag{4}$$

($u_{\vec{n}}$ is the projection of the velocity vector onto the outward normal $\vec{n}$ at the boundary, and c_r' represents known concentration values);

− on the upper surface of the parallelepiped G:

$$\frac{\partial c_r}{\partial z}=0; \tag{5}$$

− on the lower surface of the parallelepiped G:

$$\frac{\partial c_r}{\partial z}=-\varepsilon_r c_r. \tag{6}$$

Using the methods described in [9], a transformation with a “time lag” on the time grid $\overline{\omega}_\tau=\{t_n=n\tau, n=0,1,\ldots,N_t,\ N_t\tau=T\}$ was performed, along with a transition to a new coordinate system $Oxy\theta$, $\theta\in[0;1]$ according to the formulas:

$$\theta=\frac{z-\eta}{h},\ x_\theta=x,\ y_\theta=y, \tag{7}$$

where h is the depth and η is the height of the free surface relative to the mean free surface [10].

Equation (1) is then transformed as follows:

$$\begin{gathered}
\frac{\partial c_r^n}{\partial t}+C_0c_r^n=Dc_r^n+F_r^n,\ r=1,2,3,\ t_{n-1}<t\leq t_n,\ n=1,2,\ldots,N_t,\\
C_0c_r^n\equiv\frac{1}{2}\left[u\frac{\partial c_r^n}{\partial x}+v\frac{\partial c_r^n}{\partial y}+w_r'\frac{1}{H}\frac{\partial c_r^n}{\partial\theta}+\frac{\partial\left(uc_r^n\right)}{\partial x}+\frac{\partial\left(vc_r^n\right)}{\partial y}+\frac{1}{H}\frac{\partial\left(w_r'c_r^n\right)}{\partial\theta}\right],\\
Dc_r^n=\frac{\partial}{\partial x}\left(\mu_{h,r}\frac{\partial c_r^n}{\partial x}\right)+\frac{\partial}{\partial y}\left(\mu_{h,r}\frac{\partial c_r^n}{\partial y}\right)+\frac{1}{H^2}\frac{\partial}{\partial\theta}\left(\mu_{v,r}\frac{\partial c_r^n}{\partial\theta}\right),\\
F_1^n=\left(\alpha_2c_2^{n-1}(x,y,\theta,t_{n-1})-\beta_1c_1^n\right)+\gamma_1^nc_1^n,\\
F_2^n=\left(\beta_1c_1^{n-1}(x,y,\theta,t_{n-1})-\alpha_2c_2^n\right)+\left(\alpha_3c_3^{n-1}(x,y,\theta,t_{n-1})-\beta_2c_2^n\right)+\gamma_2^n\tilde{c}_2^n,\\
F_3^n=\left(\beta_2c_2^{n-1}(x,y,\theta,t_{n-1})-\alpha_3c_3^n\right)+\gamma_3^nc_3^n.
\end{gathered} \tag{8}$$

The initial and boundary conditions (2)–(6) will be transformed as follows:

$$\begin{gathered}
c_r^1(x,y,\theta,0)=c_{r,0},\ (x,y,\theta)\in\overline{G},\\
c_r^n(x,y,\theta,t_{n-1})=c_r^{n-1}(x,y,\theta,t_{n-1}),\ n=2,\ldots,N_t,\ (x,y,\theta)\in G;
\end{gathered} \tag{9}$$

$$c_r^n=c_r',\ \text{if}\ u_{\vec{n}}<0; \tag{10}$$

$$\frac{\partial c_r^n}{\partial\vec{n}}=0,\ \text{if}\ u_{\vec{n}}\geq 0, \tag{11}$$

$$\frac{\partial c_r^n}{\partial\theta}=0; \tag{12}$$

$$\frac{\partial c_r^n}{\partial\theta}=-\varepsilon_rc_r^n. \tag{13}$$

2. Second-Order Finite Difference Scheme for the Diffusion-Convection Problem of Multifractional Suspensions at Internal Nodes. Let us assume the existence of continuous (bounded) fourth-order partial derivatives with respect to the spatial variables (x, y, θ) for the functions c_r^n, $r=1,2,3$, and continuous second-order partial derivatives with respect to the time variable t. In other words, the derivatives $\dfrac{\partial^4 c_r^n}{\partial x^4}, \dfrac{\partial^4 c_r^n}{\partial y^4}, \dfrac{\partial^4 c_r^n}{\partial \theta^4}, \dfrac{\partial^2 c_r^n}{\partial t^2}$ are continuous and thus bounded for all $(x,y,\theta)\in\overline{G}$, $t_{n-1}\le t\le t_n$, $n=1,...,N_t$. and we also assume the continuity of second-order partial derivatives: $\dfrac{\partial^2\mu_{h,r}}{\partial x^2}, \dfrac{\partial^2\mu_{h,r}}{\partial y^2}, \dfrac{\partial^2\mu_{v,r}}{\partial \theta^2}, \dfrac{\partial^2 u}{\partial x^2}, \dfrac{\partial^2 v}{\partial y^2}, \dfrac{\partial^2 w_r'}{\partial \theta^2}$.

For the approximation of the problem (8)–(13), we will use the following grids:

$$\omega=\omega_x\times\omega_y\times\omega_\theta,\ \overline{\omega}=\overline{\omega}_x\times\overline{\omega}_y\times\overline{\omega}_\theta,$$

$$\omega_x=\{x: x=ih_x;\ i=1,...,N_x-1;\ (N_x-1)h_x\equiv L_x-h_x\},$$

$$\omega_y=\{y: y=jh_y;\ j=1,...,N_y-1;\ (N_y-1)h_y\equiv L_y-h_y\},$$

$$\omega_\theta=\{\theta: \theta=kh_\theta;\ k=1,...,N_\theta-1;\ (N_\theta-1)h_\theta\equiv 1-h_\theta\},$$

$$\overline{\omega}_x=\{x: x=ih_x;\ i=0,1,...,N_x;\ N_xh_x\equiv L_x\},$$

$$\overline{\omega}_y=\{y: y=jh_y;\ j=0,1,...,N_y;\ N_yh_y\equiv L_y\},$$

$$\overline{\omega}_\theta=\{\theta: \theta=kh_\theta;\ k=0,1,...,N_\theta;\ N_\theta h_\theta\equiv 1\}.$$

Next, in the notation of the grid functions for c_r^n and F_r^n we will use a bar over them. Based on the assumptions introduced, we arrive at the approximation of equation (8):

$$\frac{\overline{c}_r^n(x_i,y_j,\theta_k)-\overline{c}_r^{n-1}(x_i,y_j,\theta_k)}{\tau}+C_0\overline{c}_r^n=D\overline{c}_r^n+\overline{F}_r^n,\ r=1,2,3,\ (x_i,y_j,\theta_k)\in\omega,\ t_n\in\overline{\omega}_\tau,$$

$$C_0\overline{c}_r^n=\frac{1}{2h_x}\left(u^n(x_i+0.5h_x,y_j,\theta_k)\overline{c}_r^n(x_i+h_x,y_j,\theta_k)-u^n(x_i-0.5h_x,y_j,\theta_k)\overline{c}_r^n(x_i-h_x,y_j,\theta_k)\right)+$$

$$+\frac{1}{2h_y}\left(v^n(x_i,y_j+0.5h_y,\theta_k)\overline{c}_r^n(x_i,y_j+h_y,\theta_k)-v^n(x_i,y_j-0.5h_y,\theta_k)\overline{c}_r^n(x_i,y_j-h_y,\theta_k)\right)+$$

$$+\frac{1}{2H(x,y)h_\theta}\left(w_r'^n(x_i,y_j,\theta_k+0.5h_\theta)\overline{c}_r^n(x_i,y_j,\theta_k+h_\theta)-w_r'^n(x_i,y_j,\theta_k-0.5h_\theta)\overline{c}_r^n(x_i,y_j,\theta_k-h_\theta)\right),$$

$$D\overline{c}_r^n=\frac{1}{h_x^2}\Big(\mu_{h,r}(x_i+0.5h_x,y_j,\theta_k)\left(\overline{c}_r^n(x_i+h_x,y_j,\theta_k)-\overline{c}_r^n(x_i,y_j,\theta_k)\right)-\mu_{h,r}(x_i-0.5h_x,y_j,\theta_k)\cdot \tag{14}$$

$$\cdot\left(\overline{c}_r^n(x_i,y_j,\theta_k)-\overline{c}_r^n(x_i-h_x,y_j,\theta_k)\right)\Big)+\frac{1}{h_y^2}\Big(\mu_{h,r}(x_i,y_j+0.5h_y,\theta_k)\left(\overline{c}_r^n(x_i,y_j+h_y,\theta_k)-\overline{c}_r^n(x_i,y_j,\theta_k)\right)-$$

$$-\mu_{h,r}(x_i,y_j-0.5h_y,\theta_k)\left(\overline{c}_r^n(x_i,y_j,\theta_k)-\overline{c}_r^n(x_i,y_j-h_y,\theta_k)\right)\Big)+\frac{1}{H^2(x_i,y_j)h_\theta^2}\Big(\mu_{v,r}(x_i,y_j,\theta_k+0.5h_\theta)\cdot$$

$$\cdot\left(\overline{c}_r^n(x_i,y_j,\theta_k+h_\theta)-\overline{c}_r^n(x_i,y_j,\theta_k)\right)-\mu_{v,r}(x_i,y_j,\theta_k-0.5h_\theta)\left(\overline{c}_r^n(x_i,y_j,\theta_k)-\overline{c}_r^n(x_i,y_j,\theta_k-h_\theta)\right)\Big),$$

$$\overline{F}_1^n=\left(\alpha_2\overline{c}_2^{n-1}(x,y,\theta,t_{n-1})-\beta_1\overline{c}_1^n\right)+\gamma_1^n\overline{c}_1^n,$$

$$\overline{F}_2^n=\left(\beta_1\overline{c}_1^{n-1}(x,y,\theta,t_{n-1})-\alpha_2\overline{c}_2^n\right)+\left(\alpha_3\overline{c}_3^{n-1}(x,y,\theta,t_{n-1})-\beta_2\overline{c}_2^n\right)+\gamma_2^n\overline{c}_2^n,$$

$$\overline{F}_3^n=\left(\beta_2\overline{c}_2^{n-1}(x,y,\theta,t_{n-1})-\alpha_3\overline{c}_3^n\right)+\gamma_3^n\overline{c}_3^n.$$

We will verify that the finite difference scheme (14) has second-order accuracy. To this end, we will substitute the exact solution $c_r^n(x_i,y_j,\theta_k)\equiv c_r(x_i,y_j,\theta_k,t_n)$, $(x_i,y_j,\theta_k)\in G$, $t_n\in\omega_\tau$, $n=0,1,...,N_t$ of the problem (3)–(8) into equation (14) and demonstrate that for the approximation error

$$\psi^n(x_i,y_j,\theta_k)=-\frac{c_r^n(x_i,y_j,\theta_k)-c_r^{n-1}(x_i,y_j,\theta_k)}{\tau}-C_0c_r^n(x_i,y_j,\theta_k)+Dc_r^n(x_i,y_j,\theta_k)+F_r^n(x_i,y_j,\theta_k) \tag{15}$$

the following relationship holds:

$$\psi^n\left(x_i,y_j,\theta_k\right)=O\left(\tau+h^2\right),\ n=0,1,\ldots,N_t,\tag{16}$$

where $h^2=h_x^2+h_y^2+h_\theta^2$.

We express the function c_r^{n-1} in a Taylor series expansion around the node $\left(x_i,y_j,\theta_k,t_n\right)$:

$$c_r^{n-1}\left(x_i,y_j,\theta_k,t_{n-1}\right)=c_r^n\left(x_i,y_j,\theta_k,t_n\right)-\frac{\partial c_r^n\left(x_i,y_j,\theta_k,t_n\right)}{\partial t}\tau+\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k,t_n\right)}{\partial t^2}\frac{\tau^2}{2}+O\left(\tau^3\right).\tag{17}$$

Using relation (17) for the first term from the left-hand side of equation (15), we find:

$$\frac{c_r^n\left(x_i,y_j,\theta_k,t_n\right)-c_r^{n-1}\left(x_i,y_j,\theta_k,t_{n-1}\right)}{\tau}=\frac{1}{\tau}\left[c_r^n\left(x_i,y_j,\theta_k,t_n\right)-\left(c_r^n\left(x_i,y_j,\theta_k,t_n\right)-\frac{\partial c_r^n\left(x_i,y_j,\theta_k,t^n\right)}{\partial t}\tau+\right.\right.$$
$$\left.\left.+\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k,t_n\right)}{\partial t^2}\frac{\tau^2}{2}+O\left(\tau^3\right)\right)\right]=\frac{\partial c_r^n\left(x_i,y_j,\theta_k,t_n\right)}{\partial t}+O(\tau).\tag{18}$$

To estimate the approximation error of the convective transport operator from equation (15) we utilize the Taylor series expansions $c_r^n,u^n,v^n,w_r'^n$ around the node $\left(x_i,y_j,\theta_k,t_n\right)$:

$$c_r^n\left(x_i\pm h_x,y_j,\theta_k\right)=c_r^n\left(x_i,y_j,\theta_k\right)\pm\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial x}h_x+\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial x^2}\frac{h_x^2}{2}+O\left(h_x^3\right),\tag{19}$$

$$c_r^n\left(x_i,y_j\pm h_y,\theta_k\right)=c_r^n\left(x_i,y_j,\theta_k\right)\pm\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial y}h_x+\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial y^2}\frac{h_y^2}{2}+O\left(h_y^3\right),\tag{20}$$

$$c_r^n\left(x_i,y_j,\theta_k\pm h_\theta\right)=c_r^n\left(x_i,y_j,\theta_k\right)\pm\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial \theta}h_\theta+\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial \theta^2}\frac{h_\theta^2}{2}+O\left(h_\theta^3\right),\tag{21}$$

$$u^n\left(x_i+0.5h_x,y_j,\theta_k\right)+u^n\left(x_i-0.5h_x,y_j,\theta_k\right)=2u^n\left(x_i,y_j,\theta_k\right)+O\left(h_x^2\right),\tag{22}$$

$$u^n\left(x_i+0.5h_x,y_j,\theta_k\right)-u^n\left(x_i-0.5h_x,y_j,\theta_k\right)=\frac{\partial u^n\left(x_i,y_j,\theta_k\right)}{\partial x}h_x+O\left(h_x^3\right),\tag{23}$$

$$v^n\left(x_i,y_j+0.5h_y,\theta_k\right)+v^n\left(x_i,y_j-0.5h_y,\theta_k\right)=2v^n\left(x_i,y_j,\theta_k\right)+O\left(h_y^2\right),\tag{24}$$

$$v^n\left(x_i,y_j+0.5h_y,\theta_k\right)-v^n\left(x_i,y_j-0.5h_y,\theta_k\right)=\frac{\partial v^n\left(x_i,y_j,\theta_k\right)}{\partial y}h_y+O\left(h_y^3\right),\tag{25}$$

$$w_r'^n\left(x_i,y_j,\theta_k+0.5h_\theta\right)+w_r'^n\left(x_i,y_j,\theta_k-0.5h_\theta\right)=2w_r'^n\left(x_i,y_j,\theta_k\right)+O\left(h_\theta^2\right),\tag{26}$$

$$w_r'^n\left(x_i,y_j,\theta_k+0,5h_\theta\right)+w_r'^n\left(x_i,y_j,\theta_k-0,5h_\theta\right)=2w_r'^n\left(x_i,y_j,\theta_k\right)+O\left(h_\theta^2\right),\tag{27}$$

By substituting the corresponding expressions (19)–(27) into expression (15) for $C_0c_r^n$ we obtain:

$$C_0c_r^n=\frac{1}{2}\frac{\partial u^n\left(x_i,y_j,\theta_k\right)}{\partial x}c_r^n\left(x_i,y_j,\theta_k\right)+u^n\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial x}+\frac{1}{2}\frac{\partial v^n\left(x_i,y_j,\theta_k\right)}{\partial y}c_r^n\left(x_i,y_j,\theta_k\right)+v^n\left(x_i,y_j,\theta_k\right)\cdot$$
$$\cdot\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial y}+\frac{1}{2H\left(x_i,y_j\right)}\frac{\partial w_r'^n\left(x_i,y_j,\theta_k\right)}{\partial \theta}c_r^n\left(x_i,y_j,\theta_k\right)+\frac{1}{H\left(x_i,y_j\right)}w_r'^n\left(x_i,y_j,\theta_k\right)\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial \theta}+$$
$$+O\left(h_x^2+h_y^2+h_\theta^2\right).\tag{28}$$

To estimate the approximation error of the diffusion transport operator from equation (15), we utilize $c_r^n,\mu_{h,r},\mu_{v,r}$ in the Taylor series expansions around the point $\left(x_i,y_j,\theta_k\right)$:

$$c_r^n\left(x_i\pm h_x,y_j,\theta_k\right)=c_r^n\left(x_i,y_j,\theta_k\right)\pm\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial x}h_x+\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial x^2}\frac{h_x^2}{2}\pm\frac{\partial^3 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial x^3}\frac{h_x^3}{6}+O\left(h_x^4\right),\tag{29}$$

$$c_r^n\left(x_i,y_j\pm h_y,\theta_k\right)=c_r^n\left(x_i,y_j,\theta_k\right)\pm\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial y}h_y+\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial y^2}\frac{h_y^2}{2}\pm\frac{\partial^3 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial y^3}\frac{h_y^3}{6}+O\left(h_y^4\right),\tag{30}$$

$$c_r^n\left(x_i,y_j,\theta_k\pm h_\theta\right)=c_r^n\left(x_i,y_j,\theta_k\right)\pm\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial\theta}h_\theta+\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial\theta^2}\frac{h_\theta^2}{2}\pm\frac{\partial^3 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial\theta^3}\frac{h_\theta^3}{6}+O\left(h_\theta^4\right), \tag{31}$$

$$\mu_{h,r}\left(x_i+0.5h_x,y_j,\theta_k\right)+\mu_{h,r}\left(x_i-0.5h_x,y_j,\theta_k\right)=2\mu_{h,r}\left(x_i,y_j,\theta_k\right)+O\left(h_x^2\right), \tag{32}$$

$$\mu_{h,r}\left(x_i+0.5h_x,y_j,\theta_k\right)-\mu_{h,r}\left(x_i-0.5h_x,y_j,\theta_k\right)=\frac{\partial\mu_{h,r}\left(x_i,y_j,\theta_k\right)}{\partial x}h_x+O\left(h_x^3\right), \tag{33}$$

$$\mu_{h,r}\left(x_i,y_j+0.5h_y,\theta_k\right)+\mu_{h,r}\left(x_i,y_j-0.5h_y,\theta_k\right)=2\mu_{h,r}\left(x_i,y_j,\theta_k\right)+O\left(h_y^2\right), \tag{34}$$

$$\mu_{h,r}\left(x_i,y_j+0.5h_y,\theta_k\right)-\mu_{h,r}\left(x_i,y_j-0.5h_y,\theta_k\right)=\frac{\partial\mu_{h,r}\left(x_i,y_j,\theta_k\right)}{\partial y}h_y+O\left(h_y^3\right), \tag{35}$$

$$\mu_{v,r}\left(x_i,y_j,\theta_k+0.5h_\theta\right)+\mu_{v,r}\left(x_i,y_j,\theta_k-0.5h_\theta\right)=2\mu_{v,r}\left(x_i,y_j,\theta_k\right)+O\left(h_\theta^2\right), \tag{36}$$

$$\mu_{v,r}\left(x_i,y_j,\theta_k+0.5h_\theta\right)-\mu_{v,r}\left(x_i,y_j,\theta_k-0.5h_\theta\right)=\frac{\partial\mu_{v,r}\left(x_i,y_j,\theta_k\right)}{\partial\theta}h_\theta+O\left(h_\theta^3\right). \tag{37}$$

By substituting the corresponding expressions from equations (29)–(37) into the expression for Dc_r^n we obtain:

$$\begin{gathered}Dc_r^n=\frac{\partial\mu_{h,r}\left(x_i,y_j,\theta_k\right)}{\partial x}\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial x}+\mu_{h,r}\left(x_i,y_j,\theta_k\right)\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial x^2}+\frac{\partial\mu_{h,r}\left(x_i,y_j,\theta_k\right)}{\partial y}\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial y}+\\+\mu_{h,r}\left(x_i,y_j,\theta_k\right)\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial y^2}+\frac{1}{H^2\left(x_i,y_j\right)}\left(\frac{\partial\mu_{h,r}\left(x_i,y_j,\theta_k\right)}{\partial\theta}\frac{\partial c_r^n\left(x_i,y_j,\theta_k\right)}{\partial\theta}+\mu_{h,r}\left(x_i,y_j,\theta_k\right)\cdot\right.\\\left.\cdot\frac{\partial^2 c_r^n\left(x_i,y_j,\theta_k\right)}{\partial\theta^2}\right)+O\left(h_x^2+h_y^2+h_\theta^2\right).\end{gathered} \tag{38}$$

From the equalities (18), (28), and (38), it follows that the overall order of the approximation error of the finite difference scheme (14) at the grid nodes $\overline{\omega}_\tau\times\omega$ is equal to $O\left(\tau+h^2\right)$, $h^2=h_x^2+h_y^2+h_\theta^2$.

It is important to note that the initial condition (9) is set exactly on the grid $\overline{\omega}_\tau\times\omega$.

3. Second-Order Finite Difference Scheme for the Diffusion-Convection Problem of Multifractional Suspensions at Boundary Nodes. We will assume the existence and continuity of the derivatives $\frac{\partial^4 c_r^n}{\partial x^4},\frac{\partial^4 c_r^n}{\partial y^4},\frac{\partial^4 c_r^n}{\partial\theta^4},\frac{\partial^2 c_r^n}{\partial t^2}$, r = 1,2,3 as well as the continuity of the second-order partial derivatives: $\frac{\partial^2 u}{\partial x^2},\frac{\partial^2 v}{\partial y^2},\frac{\partial^2 w_r'}{\partial\theta^2}$, $\frac{\partial^2\mu_{h,r}}{\partial x^2},\frac{\partial^2\mu_{h,r}}{\partial y^2},\frac{\partial^2\mu_{v,r}}{\partial\theta^2}$ Additionally, we assume the existence and continuity of mixed partial derivatives. We will consider that the following conditions are satisfied:

$$\frac{\partial^2 c_r^n}{\partial x\partial t},\frac{\partial^2 c_r^n}{\partial y\partial t},\frac{\partial^2 c_r^n}{\partial\theta\partial t},\frac{\partial^4 c_r^n}{\partial t\partial x^3},\frac{\partial^4 c_r^n}{\partial t\partial y^3},\frac{\partial^4 c_r^n}{\partial t\partial\theta^3},\frac{\partial^4 c_r^n}{\partial y\partial x^3},\frac{\partial^4 c_r^n}{\partial\theta\partial x^3},\frac{\partial^4 c_r^n}{\partial x\partial y^3},\frac{\partial^4 c_r^n}{\partial\theta\partial y^3},\frac{\partial^4 c_r^n}{\partial x\partial\theta^3},\frac{\partial^4 c_r^n}{\partial y\partial\theta^3},$$

$$\frac{\partial^5 c_r^n}{\partial y^2\partial x^3},\frac{\partial^5 c_r^n}{\partial\theta^2\partial x^3},\frac{\partial^5 c_r^n}{\partial x^2\partial y^3},\frac{\partial^5 c_r^n}{\partial\theta^2\partial y^3},\frac{\partial^5 c_r^n}{\partial x^2\partial\theta^3},\frac{\partial^5 c_r^n}{\partial y^2\partial\theta^3},$$

$$\frac{\partial^2\mu_{h,r}}{\partial x\partial y},\frac{\partial^2\mu_{h,r}}{\partial x\partial\theta},\frac{\partial^2\mu_{h,r}}{\partial y\partial x},\frac{\partial^2\mu_{h,r}}{\partial\theta\partial x},\frac{\partial^2\mu_{v,r}}{\partial y\partial\theta},\frac{\partial^2 v^n}{\partial x\partial y},\frac{\partial^2 v^n}{\partial\theta\partial y},\frac{\partial^2 v^n}{\partial y\partial x},\frac{\partial^2 v^n}{\partial y\partial\theta},\frac{\partial^2 w_r'^n}{\partial x\partial\theta},\frac{\partial^2 w_r'^n}{\partial y\partial\theta},\frac{\partial^2 w_r'^n}{\partial\theta\partial x},\frac{\partial^2 w_r'^n}{\partial\theta\partial y}.$$

We will assume that the following conditions are satisfied:

$$k_{11}\le\frac{h_x}{h_y}\le k_{12},\;k_{21}\le\frac{h_\theta}{h_x}\le k_{22},\;k_{31}\le\frac{h_\theta}{h_y}\le k_{32}, \tag{39}$$

$k_{11}, k_{12}, k_{21}, k_{22}, k_{31}, k_{32}$ represents some positive constants.

To approximate the boundary conditions, we introduce an extended grid:

$$\overline{\omega}^+=\left\{\left(x_i,y_j,\theta_k\right),i=-1,0,\ldots,N_x+1,j=-1,0,\ldots,N_y+1,k=-1,0,\ldots,N_\theta+1,\right.$$

$$\left.x_i=ih_x;y_j=jh_y;\theta_k=kh_\theta;N_xh_x=L_x;N_yh_y=L_y;N_\theta h_\theta=1\right\}.$$

For the nodes of the $\overline{\omega}^{+}\backslash\overline{\omega}$ we assume the values of the components of the velocity vector are equal to zero:

$$\overline{c}_r^n\left(x_i,y_j,\theta_k\right)=0, \text{ if } \left(x_i,y_j,\theta_k\right)\in\overline{\omega}^{+}\backslash\overline{\omega} \tag{40}$$

furthermore, we consider the values of the components of the velocity vector of the water medium and the hydraulic diameter of the particles in the suspension at the grid nodes $\overline{\omega}^{+}\backslash\overline{\omega}$ with fractional indices to be known: $u^n\left(-0.5h_x,y_j,\theta_k\right)$, $u^n\left(L_x+0.5h_x,y_j,\theta_k\right)$, $v^n\left(x_i,-0.5h_y,\theta_k\right)$, $v^n\left(x_i,L_y+0.5h_y,\theta_k\right)$, $w_r'^n\left(x_i,y_j,-0.5h_\theta\right)$, $w_r'^n\left(x_i,y_j,1+0.5h_\theta\right)$.

The boundary conditions (10) are approximated as follows:

$$\begin{cases}\overline{c}_r^n\left(0,y_j,\theta_k\right)=c_r', \text{ if } u^n\left(0.5h_x,y_j,\theta_k\right)+u^n\left(-0.5h_x,y_j,\theta_k\right)>0,\\ \overline{c}_r^n\left(L_x,y_j,\theta_k\right)=c_r', \text{ if } u^n\left(L_x-0.5h_x,y_j,\theta_k\right)+u^n\left(L_x+0.5h_x,y_j,\theta_k\right)<0, \left(x_i,y_j,\theta_k\right)\in\overline{\omega}^{+};\end{cases}$$
$$\begin{cases}\overline{c}_r^n\left(x_i,0,\theta_k\right)=c_r', \text{ if } v^n\left(x_i,0.5h_y,\theta_k\right)+v^n\left(x_i,-0.5h_y,\theta_k\right)>0,\\ \overline{c}_r^n\left(x_i,L_y,\theta_k\right)=c_r', \text{ if } v^n\left(x_i,L_y-0.5h_y,\theta_k\right)+v^n\left(x_i,L_y+0.5h_y,\theta_k\right)<0, \left(x_i,y_j,\theta_k\right)\in\overline{\omega}^{+}.\end{cases} \tag{41}$$

In the case of flows on the lateral surfaces of the domain G, that coincide in direction with the external normals to the surfaces, i. e., when the conditions are met:

$$\begin{gathered}u^n\left(0.5h_x,y_j,\theta_k\right)+u^n\left(-0.5h_x,y_j,\theta_k\right)<0,\\ u^n\left(L_x-0.5h_x,y_j,\theta_k\right)+u^n\left(L_x+0.5h_x,y_j,\theta_k\right)>0, \left(x_i,y_j,\theta_k\right)\in\overline{\omega}^{+};\\ v^n\left(x_i,0.5h_y,\theta_k\right)+v^n\left(x_i,-0.5h_y,\theta_k\right)<0,\\ v^n\left(x_i,L_y-0.5h_y,\theta_k\right)+v^n\left(x_i,L_y+0.5h_y,\theta_k\right)>0, \left(x_i,y_j,\theta_k\right)\in\overline{\omega}^{+}\end{gathered} \tag{42}$$

Neumann boundary conditions are applicable.

Let us proceed to the construction of the difference scheme for the case when condition (11) is satisfied.

The condition (11) is equivalent to the following in case of $x_i = 0$:

$$\frac{\partial\overline{c}_r^n\left(0,y_j,\theta_k\right)}{\partial x}=0. \tag{43}$$

On the grid $\overline{\omega}^{+}$ the node   is internal (Fig. 1).

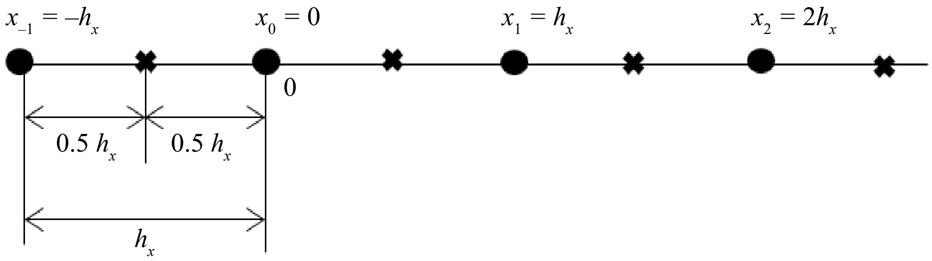

Fig. 1. Construction of the extended grid at the left end of the segment $0 \le x_i \le L_x$

The difference scheme at the nodes $\left(0,y_j,\theta_k\right)$ will be written as follows:

$$\begin{gathered}\frac{\overline{c}_r^n\left(0,y_j,\theta_k\right)-\overline{c}_r^{n-1}\left(0,y_j,\theta_k\right)}{\tau}+\frac{1}{2h_x}\left(u^n\left(0.5h_x,y_j,\theta_k\right)\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-u^n\left(-0.5h_x,y_j,\theta_k\right)\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)\right)+\\ +\frac{1}{2h_y}\left(v^n\left(0,y_j+0.5h_y,\theta_k\right)\overline{c}_r^n\left(0,y_j+h_y,\theta_k\right)-v^n\left(0,y_j-0.5h_y,\theta_k\right)\left(\overline{c}_r^n\left(0,y_j-h_y,\theta_k\right)\right)+\frac{1}{2H\left(0,y_j\right)h_\theta}\cdot\\ \cdot\left(w_r'^n\left(0,y_j,\theta_k+0.5h_\theta\right)\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-w_r'^n\left(0,y_j,\theta_k-0.5h_\theta\right)\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)\right)=\frac{1}{h_x^2}\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)\cdot\right.\\ \left.\cdot\left(\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\overline{c}_r^n\left(0,y_j,\theta_k\right)\right)-\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)\left(\overline{c}_r^n\left(0,y_j,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)\right)\right)+\frac{1}{h_y^2}\cdot\end{gathered} \tag{44}$$

$$\cdot\left(\mu_{h,r}\left(0,y_j+0.5h_y,\theta_k\right)\left(\overline{c}_r^n\left(0,y_j+h_y,\theta_k\right)-\overline{c}_r^n\left(0,y_j,\theta_k\right)\right)-\mu_{h,r}\left(0,y_j-0.5h_y,\theta_k\right)\left(\overline{c}_r^n\left(0,y_j,\theta_k\right)-\right.\right.$$

$$\left.\left.-\overline{c}_r^n\left(0,y_j-h_y,\theta_k\right)\right)\right)+\frac{1}{H^2\left(0,y_j\right)h_\theta^2}\left(\mu_{v,r}\left(0,y_j,\theta_k+0.5h_\theta\right)\left(\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-\overline{c}_r^n\left(0,y_j,\theta_k\right)\right)-\right.$$

$$\left.-\mu_{v,r}\left(0,y_j,\theta_k-0.5h_\theta\right)\left(\overline{c}_r^n\left(0,y_j,\theta_k\right)-\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)\right)\right)+\overline{F}_r^n,$$

$$\left(0,y_j,\theta_k\right)\in\overline{\omega}^+,\ r=1,2,3,\ n=1,\dots,N_t.$$

In our reasoning, we focus on the approximation of the given boundary condition using the central difference formula and the exclusion of values at the fictitious node $\left(-h_x,y_j,\theta_k\right)$ from the resulting expression and equation (44). The functions $\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)$ will be included in the expressions

$$\frac{1}{2h_x}\left(u^n\left(0.5h_x,y_j,\theta_k\right)\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-u^n\left(-0.5h_x,y_j,\theta_k\right)\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)\right)$$

$$\frac{1}{h_x^2}\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)\left(\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\overline{c}_r^n\left(0,y_j,\theta_k\right)\right)-\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)\left(\overline{c}_r^n\left(0,y_j,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)\right)\right),$$

which we will denote as $C_0\overline{c}_r^n\big|_{x_i=0}$ and $D\overline{c}_r^n\big|_{x_i=0}$.

We will write condition (43) as follows:

$$\frac{\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)}{2h_x}=0 \tag{45}$$

and from this, we obtain:

$$\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)=\overline{c}_r^n\left(h_x,y_j,\theta_k\right). \tag{46}$$

Substituting the value $\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)$ obtained from formula (46) into the expression for $C_0\overline{c}_r^n\big|_{x_i=0}$, we find:

$$C_0\overline{c}_r^n\big|_{x_i=0}\cong\frac{1}{2h_x}\left(u^n\left(0.5h_x,y_j,\theta_k\right)-u^n\left(-0.5h_x,y_j,\theta_k\right)\right)\overline{c}_r^n\left(h_x,y_j,\theta_k\right). \tag{47}$$

Preliminary calculations showed that when using equality (45), the approximation error of the expression for $C_0\overline{c}_r^n\big|_{x_i=0}$ will be $O(h^2)$, while the expression for $D\overline{c}_r^n\big|_{x_i=0}$ will be $O(h_x)$. To determine the overall order of the approximation error $O(h^2)$ of the difference scheme, a different approach will be proposed for the operator $D\overline{c}_r^n\big|_{x_i=0}$.

By expanding the functions $c_r^n\left(\pm h_x,y_j,\theta_k\right)$ in a Taylor series around the point $\left(0,y_j,\theta_k\right)$ we obtain:

$$c_r^n\left(\pm h_x,y_j,\theta_k\right)=c_r^n\left(0,y_j,\theta_k\right)\pm\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial x}h_x+\frac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial x^2}\frac{h_x^2}{2}\pm\frac{\partial^3 c_r^n\left(0,y_j,\theta_k\right)}{\partial x^3}\frac{h_x^3}{6}+ \tag{48}$$

$$+\frac{\partial^4 c_r^n\left(0,y_j,\theta_k\right)}{\partial x^4}\frac{h_x^4}{24}+O\left(h_x^5\right).$$

Using relation (48), we will explicitly write the leading term of the residual:

$$\frac{\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)}{2h_x}=\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial x}+\frac{\partial^3 c_r^n\left(0,y_j,\theta_k\right)}{\partial x^3}\frac{h_x^2}{6}+O\left(h_x^4\right).$$

The last expression, taking into account the condition $\dfrac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial x}=0$ can be written as:

$$\frac{\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)}{2h_x}=\frac{h_x^2}{6}\frac{\partial^3 c\left(0,y_j,\theta_k\right)}{\partial x^3}+O\left(h_x^4\right). \tag{49}$$

Using equality (49), we will find the value of the function c_r^{n-1} at the fictitious node $(-h_x,y,\theta)$ from the expression:

$$\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)=\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\frac{h_x^3}{3}\frac{\partial^3 c\left(0,y_j,\theta_k\right)}{\partial x^3}+O\left(h_x^5\right). \tag{50}$$

Further reasoning will focus on the approximation of the derivative $\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x^3}$.

We will differentiate both sides of equation (8) with respect to the variable and express the derivative $\frac{\partial^3 \tilde{c}_r^n}{\partial x^3}$ from the resulting equality. Next, we will take the limit as $x \to 0$ and considering that $\frac{\partial c_r^n(0,y_j,\theta_k)}{\partial x}=0$, we find:

$$\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x^3}=\frac{1}{\mu_{h,r}(0,y_j,\theta_k)}\Bigg[\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x\partial t}+u^n(0,y_j,\theta_k)\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}+\frac{\partial v^n(0,y_j,\theta_k)}{\partial x}\frac{\partial c_r^n(0,y_j,\theta_k)}{\partial y}+$$
$$+v^n(0,y_j,\theta_k)\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x\partial y}+\frac{1}{H(0,y_j)}\frac{\partial w_r'^n(0,y_j,\theta_k)}{\partial x}\frac{\partial c_r^n(0,y_j,\theta_k)}{\partial \theta}+\frac{1}{H(0,y_j)}w_r'^n(0,y_j,\theta_k)\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x\partial\theta}+$$
$$+\frac{1}{2}\frac{\partial^2 u^n(0,y_j,\theta_k)}{\partial x^2}c_r^n(0,y_j,\theta_k)+\frac{1}{2}\frac{\partial^2 v^n(0,y_j,\theta_k)}{\partial x\partial y}c_r^n(0,y_j,\theta_k)+\frac{1}{2H(0,y_j,)}\frac{\partial^2 w_r'^n(0,y_j,\theta_k)}{\partial x\partial\theta}c_r^n(0,y_j,\theta_k)-$$
$$-\frac{\partial \mu_{h,r}(0,y_j,\theta_k)}{\partial x}\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}-\frac{\partial \mu_{h,r}(0,y_j,\theta_k)}{\partial x}\frac{\partial^2 \tilde{c}_r^n(0,y_j,\theta_k)}{\partial y^2}-\mu_{h,r}(0,y_j,\theta_k)\frac{\partial^3 \tilde{c}_r^n}{\partial x\partial y^2}-\frac{1}{H^2(0,y_j)}\cdot \tag{51}$$
$$\cdot\left(\frac{\partial^2 \mu_{v,r}(0,y_j,\theta_k)}{\partial x\partial\theta}\frac{\partial c_r^n(0,y_j,\theta_k)}{\partial\theta}+\frac{\partial \mu_{v,r}(0,y_j,\theta_k)}{\partial\theta}\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x\partial\theta}+\frac{\partial \mu_{v,r}(0,y_j,\theta_k)}{\partial x}\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial\theta^2}+\right.$$
$$\left.+\mu_{v,r}(0,y_j,\theta_k)\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x\partial\theta^2}\right)\Bigg],\ r=1,2,3,\ (0,y_j,\theta_k)\in\bar{G}.$$

It is evident that the equality holds:

$$\frac{\partial F_1^n(0,y_j,\theta_k)}{\partial x}=\frac{\partial F_2^n(0,y_j,\theta_k)}{\partial x}=\frac{\partial F_3^n(0,y_j,\theta_k)}{\partial x}=0.$$

For the reader's convenience, we will approximate the expression in parentheses from the right side of expression (51) for each term separately. Initially, we note that for the coefficient $\frac{1}{\mu_{h,r}(0,y_j,\theta_k)}$, which stands before this parentheses, we will use the expression:

$$\frac{1}{\mu_{h,r}(0,y_j,\theta_k)}=\frac{2}{\mu_{h,r}(0.5h_x,y_j,\theta_k)+\mu_{h,r}(-0.5h_x,y_j,\theta_k)}.$$

Let us consider the derivative $\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x\partial t}$. For it, we have:

$$\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x\partial t}=\frac{1}{2h_x}\left(\frac{\partial \bar{c}_r^n(h_x,y_j,\theta_k,t_n)}{\partial t}-\frac{\partial \bar{c}_r^n(-h_x,y_j,\theta_k,t_n)}{\partial t}\right)+O(h_x^2)=\frac{1}{2h_x}\cdot$$
$$\cdot\left(\frac{\bar{c}_r^n(h_x,y_j,\theta_k,t_n+\tau)-\bar{c}_r^n(h_x,y_j,\theta_k,t^n-\tau)}{2\tau}-\frac{\bar{c}_r^n(-h_x,y_j,\theta_k,t^n+\tau)-\bar{c}_r^n(-h_x,y_j,\theta_k,t^n-\tau)}{2\tau}+O(\tau^2)\right)+$$
$$+O(h_x^2)=\frac{1}{2h_x}\left(\frac{\bar{c}_r^n(h_x,y_j,\theta_k,t^n+\tau)-\bar{c}_r^n(-h_x,y_j,\theta_k,t^n+\tau)}{2h_x}-\frac{\bar{c}_r^n(h_x,y_j,\theta_k,t^n-\tau)-\bar{c}_r^n(-h_x,y_j,\theta_k,t^n-\tau)}{2h_x}\right)+ \tag{53}$$
$$+\frac{1}{2h_x}O(\tau^2)+O(h_x^2).$$

Using equality (50), the relationships can be written as:

$$\frac{\bar{c}_r^n(h_x,y_j,\theta_k,t_n\pm\tau)-\bar{c}_r^n(-h_x,y_j,\theta_k,t_n\pm\tau)}{2h_x}=\frac{h_x^2}{6}\frac{\partial^3 c_r^n(0,y_j,\theta_k,t_n\pm\tau)}{\partial x^3}+O(h_x^4). \tag{54}$$

With the help of equality (54), we will transform equality (53):

$$\frac{\partial^2 c_r^n}{\partial x\partial t}=\frac{1}{2\tau}\frac{h_x^2}{6}\left(\frac{\partial^3 c_r^n\left(0,y_j,\theta_k,t_n+\tau\right)}{\partial x^3}-\frac{\partial^3 c_r^n\left(0,y_j,\theta_k,t_n-\tau\right)}{\partial x^3}\right)+\frac{1}{2h_x}O\left(\tau^2\right)+O\left(h_x^2\right).$$

Introducing the notation $\varphi\left(0,y_j,\theta_k,t_n\pm\tau\right)=\dfrac{\partial^3 c_r^n\left(0,y_j,\theta_k,t_n\pm\tau\right)}{\partial x^3}$, we will write the last equality as:

$$\frac{\partial^2 c_r^n}{\partial x\partial t}=\frac{h_x^2}{6}\left(\frac{\varphi\left(0,y_j,\theta_k,t_n+\tau\right)-\varphi\left(0,y_j,\theta_k,t_n-\tau\right)}{2\tau}\right)+\frac{1}{2h_x}O\left(\tau^2\right)+O\left(h_x^2\right). \quad (55)$$

From which it follows:

$$\frac{\partial^2 c_r^n}{\partial x\partial t}=\frac{h_x^2}{6}\left(\frac{\partial\varphi\left(0,y_j,\theta_k,t_n\right)}{\partial t}+O\left(\tau^2\right)\right)+\frac{1}{2h_x}O\left(\tau^2\right)+O\left(h_x^2\right).$$

In accordance with the Courant condition [11], the quantity is bounded, and we can assume that the equality holds: $\dfrac{1}{2h_x}O\left(\tau^2\right)=O\left(h_x\right)$. Taking this into account, we have:

$$\frac{\partial^2 c_r^n}{\partial x\partial t}=\frac{h_x^2}{6}\frac{\partial\varphi\left(0,y_j,\theta_k,t_n\right)}{\partial t}+O\left(h_x\right). \quad (56)$$

Considering relation (56) and the boundedness of the derivative $\dfrac{\partial\varphi\left(0,y_j,\theta_k,t_n\pm\tau\right)}{\partial t}=\dfrac{\partial^4 c_r^n\left(0,y_j,\theta_k,t_n\pm\tau\right)}{\partial t\partial x^3}$ the expression $\dfrac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial x\partial t}$ is equal to zero up to $O(h_x)$.

Next, we will show that the derivative $\dfrac{\partial^3 c_r^n\left(0,y_j,\theta_k\right)}{\partial x\partial y^2}$ when approximated with an error of $O(h_x)$ also tends to zero.

Taking into account the inequality $k_{11}\le\dfrac{h_x}{h_y}\le k_{12}$ from (39), we have:

$$\begin{aligned}
\frac{\partial^3 c_r^n\left(0,y_j,\theta_k\right)}{\partial x\partial y^2}&=\frac{1}{2h_x}\left(\frac{\partial^2\overline{c}_r^n\left(h_x,y_j,\theta_k\right)}{\partial y^2}-\frac{\partial^2\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)}{\partial y^2}\right)+O\left(h_x^2\right)=\\
&=\frac{1}{2h_x}\left(\frac{\overline{c}_r^n\left(h_x,y_j+h_y,\theta_k\right)-2\overline{c}_r^n\left(h_x,y_j,\theta_k\right)+\overline{c}_r^n\left(h_x,y_j-h_y,\theta_k\right)}{h_y^2}-\right.\\
&\left.-\frac{\overline{c}_r^n\left(-h_x,y_j+h_y,\theta_k\right)-2\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)+\overline{c}_r^n\left(-h_x,y_j-h_y,\theta_k\right)}{h_y^2}+O\left(h_y^2\right)\right)+O\left(h_x^2\right)=\\
&=\frac{1}{h_y^2}\left(\frac{\overline{c}_r^n\left(h_x,y_j+h_y,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j+h_y,\theta_k\right)}{2h_x}-2\frac{\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)}{2h_x}+\right.\\
&\left.+\frac{\overline{c}_r^n\left(h_x,y_j-h_y,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j-h_y,\theta_k\right)}{2h_x}\right)+\frac{1}{2h_x}O\left(h_y^2\right)+O\left(h_x^2\right)=\\
&=\frac{1}{h_y^2}\left(\frac{\overline{c}_r^n\left(h_x,y_j+h_y,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j+h_y,\theta_k\right)}{2h_x}-2\frac{\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j,\theta_k\right)}{2h_x}+\right.\\
&\left.+\frac{\overline{c}_r^n\left(h_x,y_j-h_y,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j-h_y,\theta_k\right)}{2h_x}\right)+O\left(h_x^2+h_y\right).
\end{aligned} \quad (57)$$

Based on equality (57), the relationship can be written as:

$$\frac{\overline{c}_r^n\left(h_x,y_j\pm h_y,\theta_k\right)-\overline{c}_r^n\left(-h_x,y_j\pm h_y,\theta_k\right)}{2h_x}=\frac{h_x^2}{6}\frac{\partial^3 c_r^n\left(0,y_j\pm h_y,\theta_k\right)}{\partial x^3}+O\left(h_x^4\right). \quad (58)$$

Using equalities (58), we will transform relationship (57):

$$\frac{\partial^3 c_r^n\left(0,y_j,\theta_k\right)}{\partial x\partial y^2}=\frac{1}{h_y^2}\frac{h_x^2}{6}\left(\frac{\partial^3 c_r^n\left(0,y_j+h_y,\theta_k\right)}{\partial x^3}-2\frac{\partial^3 c_r^n\left(0,y_j,\theta_k\right)}{\partial x^3}+\frac{\partial^3 c_r^n\left(0,y_j-h_y,\theta_k\right)}{\partial x^3}\right)+O\left(h_x^2+h_y\right).$$

Let $\varphi(0,y_j\pm h_y,\theta_k)=\dfrac{\partial^3 c_r^n(0,y_j\pm h_y,\theta_k)}{\partial x^3}$, $\varphi(0,y_j,\theta_k)=\dfrac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x^3}$. Then the last equality can be written as:

$$\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x\partial y^2}=\frac{h_x^2}{6}\left(\frac{\varphi(0,y_j+h_y,\theta_k)-2\varphi(0,y_j,\theta_k)+\varphi(0,y_j-h_y,\theta_k)}{h_y^2}\right)+O(h_x^2+h_y). \tag{59}$$

From equality (59), it follows that:

$$\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x\partial y^2}=\frac{h_x^2}{6}\left(\frac{\partial^2\varphi(0,y_j,\theta_k)}{\partial y^2}+O(h_y^2)\right)+O(h_x^2+h_y).$$

The last equality can be transformed into the form:

$$\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x\partial y^2}=\frac{h_x^2}{6}\frac{\partial^2\varphi(0,y_j,\theta_k)}{\partial y^2}+O(h_x). \tag{60}$$

Considering relation (60) and the boundedness of the derivative $\dfrac{\partial^2\varphi(0,y_j,\theta_k)}{\partial y^2}=\dfrac{\partial^5 c_r^n(0,y_j,\theta_k)}{\partial y^2\partial x^3}$ with respect to $O(h_x)$ the expression $\dfrac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x\partial y^2}$ is equal to zero.

By applying similar reasoning for the derivatives $\dfrac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x\partial\theta^2}$, $\dfrac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x\partial y}$, $\dfrac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x\partial\theta}$ it can be easily verified that when the inequalities from (40) are satisfied, these derivatives are equal to zero up to $O(h_x)$.

Let us consider the derivative $\dfrac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}$. We have:

$$\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}=\frac{\overline{c}_r^n(h_x,y_j,\theta_k)-2\overline{c}_r^n(0,y_j,\theta_k)+\overline{c}_r^n(-h_x,y_j,\theta_k)}{h_x^2}+O(h_x^2). \tag{61}$$

In the last equality, the value of the function $\overline{c}_r^n$ at the fictitious node $(-h_x,y_j,\theta_k)$ will be replaced using expression (50). We obtain

$$\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}=\frac{1}{h_x^2}\left(\overline{c}_r^n(h_x,y_j,\theta_k)-2\overline{c}_r^n(0,y_j,\theta_k)+\left(\overline{c}_r^n(h_x,y_j,\theta_k)-\frac{h_x^3}{3}\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x^3}+O(h_x^5)\right)\right)+$$

$$+O(h_x^2)=\frac{1}{h_x^2}\left(2\left(\overline{c}_r^n(h_x,y_j,\theta_k)-\overline{c}_r^n(0,y_j,\theta_k)\right)-\frac{h_x^3}{3}\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x^3}\right)+O(h_x^2)$$

or

$$\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}\cong\frac{2}{h_x^2}\left(\overline{c}_r^n(h_x,y_j,\theta_k)-\overline{c}_r^n(0,y_j,\theta_k)\right)-\frac{h_x}{3}\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x^3}. \tag{62}$$

The terms $u^n(0,y_j,\theta_k)\dfrac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}$ and $\dfrac{\partial\mu_{h,r}(0,y_j,\theta_k)}{\partial x}\dfrac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}$ from equality (51), which include the factor $\dfrac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}$, are approximated by the expressions using relation (62):

$$\begin{aligned}u^n(0,y_j,\theta_k)\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}\cong\frac{1}{2}\left(u^n(0.5h_x,y_j,\theta_k)+u^n(-0.5h_x,y_j,\theta_k)\right)\left(\frac{2}{h_x^2}\left(\overline{c}_r^n(h_x,y_j,\theta_k)-\overline{c}_r^n(0,y_j,\theta_k)\right)-\right.\\ \left.-\frac{h_x}{3}\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x^3}\right);\end{aligned} \tag{63}$$

$$\begin{aligned}\frac{\partial\mu_{h,r}(0,y_j,\theta_k)}{\partial x}\frac{\partial^2 c_r^n(0,y_j,\theta_k)}{\partial x^2}\cong\frac{1}{h_x}\left(\mu_{h,r}(0.5h_x,y_j,\theta_k)-\mu_{h,r}(-0.5h_x,y_j,\theta_k)\right)\left(\frac{2}{h_x^2}\left(\overline{c}_r^n(h_x,y_j,\theta_k)-\right.\right.\\ \left.\left.-\overline{c}_r^n(0,y_j,\theta_k)\right)-\frac{h_x}{3}\frac{\partial^3 c_r^n(0,y_j,\theta_k)}{\partial x^3}\right).\end{aligned} \tag{64}$$

When approximating the derivative $\dfrac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial y^2}$, we obtain:

$$\frac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial y^2}=\frac{\overline{c}_r^n\left(0,y_j+h_y,\theta_k\right)-2\overline{c}_r^n\left(0,y_j,\theta_k\right)+\overline{c}_r^n\left(0,y_j-h_y,\theta_k\right)}{h_y^2}+O\left(h_y^2\right). \tag{65}$$

Approximating the term $\dfrac{\partial\mu_{h,r}\left(0,y_j,\theta_k\right)}{\partial x}\dfrac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial y^2}$ from equality (51), which includes the factor $\dfrac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial y^2}$, using relation (65), we get:

$$\begin{gathered}\frac{\partial\mu_{h,r}\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial y^2}\cong\frac{1}{h_x h_y^2}\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k\right)\right)\left(\overline{c}_r^n\left(0,y_j+h_y,\theta_k\right)-\right.\\ \left.-2\overline{c}_r^n\left(0,y_j,\theta_k\right)+\overline{c}_r^n\left(0,y_j-h_y,\theta_k\right)\right).\end{gathered} \tag{66}$$

The approximation of the form (66) is carried out with an accuracy of $O(h_x)$. Indeed, it is not difficult to verify:

$$\overline{c}_r^n\left(0,y_j+h_y,\theta_k\right)-2\overline{c}_r^n\left(0,y_j,\theta_k\right)+\overline{c}_r^n\left(0,y_j-h_y,\theta_k\right)=\frac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial y^2}h_y^2+O\left(h_y^3\right), \tag{67}$$

$$\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)-\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)=\frac{\partial\mu_{h,r}\left(0,y_j,\theta_k\right)}{\partial x}h_x+O\left(h_x^3\right). \tag{68}$$

Taking into account equalities (67) and (68) for relation (66), we obtain:

$$\begin{gathered}\frac{\partial\mu_{h,r}\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial y^2}=\frac{1}{h_x h_y^2}\left(\frac{\partial\mu_{h,r}\left(0,y_j,\theta_k\right)}{\partial x}h_x+O\left(h_x^3\right)\right)\left(\frac{\partial^2\overline{c}_r^n\left(0,y_j,\theta_k\right)}{\partial y^2}h_y^2+O\left(h_y^3\right)\right)=\\ =\frac{\partial\mu_{h,r}\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial^2\overline{c}_r^n\left(0,y_j,\theta_k\right)}{\partial y^2}+O\left(h_x\right).\end{gathered} \tag{69}$$

When approximating the derivative $\dfrac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}$, we obtain:

$$\frac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}=\frac{\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-2\overline{c}_r^n\left(0,y_j,\theta_k\right)+\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)}{h_\theta^2}+O\left(h_\theta^2\right). \tag{70}$$

Then for the term $\dfrac{1}{H^2\left(0,y_j\right)}\dfrac{\partial\mu_{v,r}\left(0,y_j,\theta_k\right)}{\partial x}\dfrac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}$ from equality (51), which includes the factor $\dfrac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}$, we find:

$$\begin{gathered}\frac{1}{H^2\left(0,y_j\right)}\frac{\partial\mu_{v,r}\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}\cong\frac{1}{h_x h_\theta^2 H^2\left(0,y_j\right)}\left(\mu_{v,r}\left(0.5h_x,y_j,\theta_k\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k\right)\right)\cdot\\ \cdot\left(\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-2\overline{c}_r^n\left(0,y_j,\theta_k\right)+\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)\right).\end{gathered} \tag{71}$$

The error of the approximation of the expression $\dfrac{1}{H^2\left(0,y_j\right)}\dfrac{\partial\mu_{v,r}\left(0,y_j,\theta_k\right)}{\partial x}\dfrac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}$ using relation (71) is $O(h_x)$. Indeed, taking into account the equality

$$\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-2\overline{c}_r^n\left(0,y_j,\theta_k\right)+\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)=2\frac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}\frac{h_\theta^2}{2}+O\left(h_\theta^3\right), \tag{72}$$

we obtain:

$$\begin{gathered}\frac{1}{H^2\left(0,y_j\right)}\frac{\partial\mu_{v,r}\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial^2 c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}=\frac{1}{h_x h_\theta^2 H^2\left(0,y_j\right)}\left(\frac{\partial\mu_{h,r}\left(0,y_j,\theta_k\right)}{\partial x}h_x+O\left(h_x^3\right)\right)\left(\frac{\partial^2\overline{c}_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}h_\theta^2+\right.\\ \left.+O\left(h_\theta^3\right)\right)=\frac{\partial\mu_{h,r}\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial^2\overline{c}_r^n\left(0,y_j,\theta_k\right)}{\partial\theta^2}+O\left(h_x\right).\end{gathered} \tag{73}$$

Next, let us consider the derivative $\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial y}$. When approximating it with central differences, we get:

$$\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial y}=\frac{\overline{c}_r^n\left(0,y_j+h_y,\theta_k\right)-\overline{c}_r^n\left(0,y_j-h_y,\theta_k\right)}{2h_y}+O\left(h_y^2\right). \tag{74}$$

Then for the term $\frac{\partial v^n\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial y}$ from equality (51), which includes the factor $\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial y}$, we find:

$$\frac{\partial v^n\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial y}\cong\frac{1}{2h_xh_y}\left(v^n\left(0.5h_x,y_j,\theta_k\right)-v^n\left(-0.5h_x,y_j,\theta_k\right)\right)\left(\overline{c}_r^n\left(0,y_j+h_y,\theta_k\right)-\right.$$
$$\left.-\overline{c}_r^n\left(0,y_j-h_y,\theta_k\right)\right). \tag{75}$$

Here

$$\frac{\partial v^n\left(0,y_j,\theta_k\right)}{\partial x}=\frac{v^n\left(0.5h_x,y_j,\theta_k\right)-v^n\left(-0.5h_x,y_j,\theta_k\right)}{h_x}+O\left(h_x^2\right). \tag{76}$$

It is not difficult to verify that the approximation error for expression $\frac{\partial v^n\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial y}$, carried out using relation (75), is $O\left(h_x^2\right)$ and the equality holds:

$$\frac{\partial v^n\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial y}=\frac{1}{2h_xh_y}\left(\frac{\partial v^n\left(0,y_j,\theta_k\right)}{\partial x}h_x+O\left(h_x^3\right)\right)\left(2\frac{\partial\overline{c}_r^n\left(0,y_j,\theta_k\right)}{\partial y}h_y+O\left(h_y^3\right)\right)=$$
$$=2\frac{\partial v^n\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial\overline{c}_r^n\left(0,y_j,\theta_k\right)}{\partial y}+O\left(h_x^2\right). \tag{77}$$

When approximating the derivative $\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta}$, we obtain:

$$\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta}=\frac{\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)}{2h_\theta}+O\left(h_\theta^2\right). \tag{78}$$

Then for the terms $\frac{1}{H\left(0,y_j\right)}\frac{\partial w_r'^n\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta}$ and $\frac{1}{H^2\left(0,y_j\right)}\frac{\partial^2\mu_{v,r}\left(0,y_j,\theta_k\right)}{\partial x\partial\theta}\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta}$ from equality (51), which include the factor $\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta}$, we find:

$$\frac{1}{H\left(0,y_j\right)}\frac{\partial w_r'^n\left(0,y_j,\theta_k\right)}{\partial x}\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta}\cong\frac{1}{2h_xh_\theta H\left(0,y_j\right)}\left(w_r'^n\left(0.5h_x,y_j,\theta_k\right)-w_r'^n\left(-0.5h_x,y_j,\theta_k\right)\right)\cdot$$
$$\cdot\left(\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)\right); \tag{79}$$

$$\frac{1}{H^2\left(0,y_j\right)}\frac{\partial^2\mu_{v,r}\left(0,y_j,\theta_k\right)}{\partial x\partial\theta}\frac{\partial c_r^n\left(0,y_j,\theta_k\right)}{\partial\theta}\cong\frac{1}{2h_xh_\theta^2H^2\left(0,y_j\right)}\left[\left(\mu_{v,r}\left(0.5h_x,y_j,\theta_k+0.5h_\theta\right)-\right.\right.$$
$$\left.-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k+0.5h_\theta\right)\right)-\left.\left(\mu_{v,r}\left(0.5h_x,y_j,\theta_k-0.5h_\theta\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k-0.5h_\theta\right)\right)\right]\cdot$$
$$\cdot\left(\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)\right). \tag{80}$$

Here, the equalities used in writing relations (79) and (80) are:

$$\frac{\partial w_r'^n\left(0,y_j,\theta_k\right)}{\partial x}=\frac{w_r'^n\left(0.5h_x,y_j,\theta_k\right)-w_r'^n\left(-0.5h_x,y_j,\theta_k\right)}{h_x}+O\left(h_x^2\right) \tag{81}$$

$$\frac{\partial^2\mu_{v,r}\left(0,y_j,\theta_k\right)}{\partial x\partial\theta}=\frac{1}{h_\theta}\left(\frac{\mu_{v,r}\left(0.5h_x,y_j,\theta_k+0.5h_\theta\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k+0.5h_\theta\right)}{h_x}-\right.$$
$$\left.-\frac{\mu_{v,r}\left(0.5h_x,y_j,\theta_k-0.5h_\theta\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k-0.5h_\theta\right)}{h_x}\right)+\frac{1}{h_x}O\left(h_\theta^2\right)+O\left(h_x^2\right). \quad (82)$$

Taking into account the inequality $k_{21}\le\frac{h_\theta}{h_x}\le k_{22}$ from (39), we can assert that the approximation of the form (80) is carried out with an error of $O(h_x)$.

When approximating $c_r^n\left(0,y_j,\theta_k\right)$, we replace it with its grid analog $\overline{c}_r^n\left(0,y_j,\theta_k\right)$.

Then for the terms $\frac{1}{2}\frac{\partial^2u^n\left(0,y_j,\theta_k\right)}{\partial x^2}c_r^n\left(0,y_j,\theta_k\right)$, $\frac{1}{2}\frac{\partial^2v^n\left(0,y_j,\theta_k\right)}{\partial x\partial y}c_r^n\left(0,y_j,\theta_k\right)$ and $\frac{1}{2H\left(0,y_j\right)}\frac{\partial^2w_r'^n\left(0,y_j,\theta_k\right)}{\partial x\partial\theta}c_r^n\left(0,y_j,\theta_k\right)$, which include the factor $\overline{c}_r^n\left(0,y_j,\theta_k\right)$, we find:

$$\frac{1}{2}\frac{\partial^2u^n\left(0,y_j,\theta_k\right)}{\partial x^2}c_r^n\left(0,y_j,\theta_k\right)\cong\frac{1}{0.5h_x^2H\left(0,y_j\right)}\left(u^n\left(0.5h_x,y_j,\theta_k\right)-2u^n\left(0,y_j,\theta_k\right)+u^n\left(-0.5h_x,y_j,\theta_k\right)\right)\cdot$$
$$\cdot\overline{c}_r^n\left(0,y_j,\theta_k\right); \quad (83)$$

$$\frac{1}{2}\frac{\partial^2v^n\left(0,y_j,\theta_k\right)}{\partial x\partial y}c_r^n\left(0,y_j,\theta_k\right)\cong\frac{1}{2h_xh_y}\left[\left(v^n\left(0.5h_x,y_j+0.5h_y,\theta_k\right)-v^n\left(-0.5h_x,y_j+0.5h_y,\theta_k\right)\right)-\right.$$
$$\left.-\left(v^n\left(0.5h_x,y_j-0.5h_y,\theta_k\right)-v^n\left(-0.5h_x,y_j-0.5h_y,\theta_k\right)\right)\right]\overline{c}_r^n\left(0,y_j,\theta_k\right). \quad (84)$$

$$\frac{1}{2H\left(0,y_j\right)}\frac{\partial^2w_r'^n\left(0,y_j,\theta_k\right)}{\partial x\partial\theta}c_r^n\left(0,y_j,\theta_k\right)\cong\frac{1}{2h_xh_\theta H\left(0,y_j\right)}\left[\left(w_r'^n\left(0.5h_x,y,\theta_k+0.5h_\theta\right)-\right.\right.$$
$$\left.-\left(w_r'^n\left(-0.5h_x,y_j,\theta_k+0.5h_\theta\right)\right)\left(w_r'^n\left(0.5h_x,y_j,\theta_k-0.5h_\theta\right)-w_r'^n\left(-0.5h_x,y_j,\theta_k-0.5h_\theta\right)\right)\right]\overline{c}_r^n\left(0,y_j,\theta_k\right). \quad (85)$$

It is evident that expression (83) is obtained with an accuracy of $O(h^2_x)$.

In writing relations (84) and (85), the equalities used are:

$$\frac{\partial^2v^n\left(0,y_j,\theta_k\right)}{\partial x\partial y}=\frac{1}{h_y}\left(\frac{v^n\left(0.5h_x,y_j+0.5h_y,\theta_k\right)-v^n\left(-0.5h_x,y_j+0.5h_y,\theta_k\right)}{h_x}-\right.$$
$$\left.-\frac{v^n\left(0.5h_x,y_j-0.5h_y,\theta_k\right)-v^n\left(-0.5h_x,y_j-0.5h_y,\theta_k\right)}{h_x}\right)+\frac{1}{h_x}O\left(h_y^2\right)+O\left(h_x^2\right), \quad (86)$$

$$\frac{\partial^2w_r'^n\left(0,y_j,\theta_k\right)}{\partial x\partial\theta}=\frac{1}{h_\theta}\left(\frac{w_r'^n\left(0.5h_x,y_j,\theta_k+0.5h_\theta\right)-w_r'^n\left(-0.5h_x,y_j,\theta_k+0.5h_\theta\right)}{h_x}-\right.$$
$$\left.-\frac{w_r'^n\left(0.5h_x,y_j-0.5h_y,\theta_k\right)-w_r'^n\left(-0.5h_x,y_j-0.5h_y,\theta_k\right)}{h_x}\right)+\frac{1}{h_x}O\left(h_\theta^2\right)+O\left(h_x^2\right). \quad (87)$$

Considering equalities (86) and (87), we obtain that the error of the approximations (84) and (85) is $O(h_x)$.

Thus, we have obtained the approximations for all terms located in the parentheses on the right side of equality (51). As a result of substituting into equality (51) the approximations carried out by expressions (63), (64), (66), (71), (75), (79), (80), (83)–(85) with an error of $O(h_x)$ ((or higher), we obtain:

$$\frac{\partial^3\overline{c}_r^n\left(0,y_j,\theta_k\right)}{\partial x^3}=\vartheta_1+O\left(h_x\right), \quad (88)$$

where

$$\vartheta_1=\vartheta_{10}\left(\vartheta_{11}\overline{c}\left(h_x,y_j,\theta_k\right)+\vartheta_{12}\overline{c}\left(0,y_j+h_y,\theta_k\right)+\vartheta_{13}\overline{c}\left(0,y_j-h_y,\theta_k\right)+\vartheta_{14}\overline{c}\left(0,y_j,\theta_k+h_\theta\right)+\vartheta_{15}\overline{c}\left(0,y_j,\theta_k-h_\theta\right)+\right.$$
$$\left.+\vartheta_{16}\overline{c}\left(0,y_j,\theta_k\right)\right),$$

$$\vartheta_{10}=\frac{3\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)+\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)\right)}{4\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)+2\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)-\left(u^n\left(0.5h_x,y_j,\theta_k\right)+u^n\left(-0.5h_x,y_j,\theta_k\right)\right)h_x};$$

$$\vartheta_{11}=\frac{1}{h_x^2}\left(u^n\left(0.5h_x,y_j,\theta_k\right)+u^n\left(-0.5h_x,y_j,\theta_k\right)\right)-\frac{2}{h_x^3}\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)-\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)\right);$$

$$\vartheta_{12}=\frac{1}{2h_xh_y}\left(v^n\left(0.5h_x,y_j,\theta_k\right)-v^n\left(-0.5h_x,y_j,\theta_k\right)\right)-\frac{1}{h_xh_y^2}\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k\right)\right);$$

$$\vartheta_{12}=\frac{1}{2h_xh_y}\left(v^n\left(0.5h_x,y_j,\theta_k\right)-v^n\left(-0.5h_x,y_j,\theta_k\right)\right)-\frac{1}{h_xh_y^2}\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k\right)\right);$$

$$\vartheta_{13}=-\frac{1}{2h_xh_y}\left(v^n\left(0.5h_x,y_j,\theta_k\right)-v^n\left(-0.5h_x,y_j,\theta_k\right)\right)-\frac{1}{h_xh_y^2}\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k\right)\right);$$

$$\vartheta_{14}=\frac{1}{2h_xh_\theta H\left(0,y_j\right)}\left(w_r'^n\left(0.5h_x,y_j,\theta_k\right)-w_r'^n\left(-0.5h_x,y_j,\theta_k\right)\right)-\frac{1}{h_xh_\theta^2H^2\left(0,y_j\right)}\left(\mu_{v,r}\left(0.5h_x,y_j,\theta_k\right)-\right.$$

$$\left.-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k\right)\right)-\frac{1}{2h_xh_\theta^2H^2\left(0,y_j\right)}\left(\mu_{v,r}\left(0.5h_x,y_j,\theta_k+0.5h_\theta\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k+0.5h_\theta\right)-\right.$$

$$\left.-\mu_{v,r}\left(0.5h_x,y_j,\theta_k-0.5h_\theta\right)+\mu_{v,r}\left(-0.5h_x,y_j,\theta_k-0.5h_\theta\right)\right);$$

$$\vartheta_{15}=-\frac{1}{2h_xh_\theta H\left(0,y_j\right)}\left(w_r'^n\left(0.5h_x,y_j,\theta_k\right)-w_r'^n\left(-0.5h_x,y_j,\theta_k\right)\right)-\frac{1}{h_xh_\theta^2H^2\left(0,y_j\right)}\left(\mu_{v,r}\left(0.5h_x,y_j,\theta_k\right)-\right.$$

$$\left.-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k\right)\right)+\frac{1}{2h_xh_\theta^2H^2\left(0,y_j\right)}\left(\mu_{v,r}\left(0.5h_x,y_j,\theta_k+0.5h_\theta\right)-\mu_{v,r}\left(-0.5h_x,y_j,\theta_k+0.5h_\theta\right)-\right.$$

$$\left.-\mu_{v,r}\left(0.5h_x,y_j,\theta_k-0.5h_\theta\right)+\mu_{v,r}\left(-0.5h_x,y_j,\theta_k-0.5h_\theta\right)\right);$$

$$\vartheta_{16}=-\frac{1}{h_x^2}\left(u^n\left(0.5h_x,y_j,\theta_k\right)+u^n\left(-0.5h_x,y_j,\theta_k\right)\right)+\frac{2}{h_x^2H\left(0,y_j\right)}\left(u^n\left(0.5h_x,y_j,\theta_k\right)-\right.$$

$$\left.-2u^n\left(0,y_j,\theta_k\right)+u^n\left(-0.5h_x,y_j,\theta_k\right)\right)+\frac{1}{2h_xh_y}\left(v^n\left(0.5h_x,y_j+0.5h_y,\theta_k\right)-v^n\left(-0.5h_x,y_j+0.5h_y,\theta_k\right)-\right.$$

$$\left.-v^n\left(0.5h_x,y_j-0.5h_y,\theta_k\right)+v^n\left(-0.5h_x,y_j-0.5h_y,\theta_k\right)\right)+\frac{1}{2h_xh_\theta H\left(0,y_j\right)}\left(w_r'^n\left(0.5h_x,y_j,\theta_k+0.5h_\theta\right)-\right.$$

$$\left.-w_r'^n\left(-0.5h_x,y_j,\theta_k+0.5h_\theta\right)-w_r'^n\left(0.5h_x,y_j,\theta_k-0.5h_\theta\right)+w_r'^n\left(-0.5h_x,y_j,\theta_k-0.5h_\theta\right)\right)+$$

$$+2\left(\frac{1}{h_x^3}+\frac{1}{h_xh_y^2}+\frac{1}{h_xh_\theta^2H^2\left(0,y_j\right)}\right)\left(\mu_{h,r}\left(0.5h_x,y,\theta\right)-\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)\right).$$

Using equality (88) for $D\overline{c}_r^n\big|_{x_i=0}$, we can construct the expression:

$$D\overline{c}_r^n\big|_{x_i=0}\cong\frac{1}{h_x^2}\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)+\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)\right)\left(\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\overline{c}_r^n\left(0,y_j,\theta_k\right)\right)-$$
$$-\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)\frac{h_x}{3}\vartheta_1. \tag{89}$$

Ultimately, the difference scheme (44), taking into account relations (47) and (89), will take the form:

$$\frac{\overline{c}_r^n-\overline{c}_r^{n-1}}{\tau}+\frac{1}{2h_x}\left(u^n\left(0.5h_x,y_j,\theta_k\right)-u^n\left(-0.5h_x,y_j,\theta_k\right)\right)\overline{c}_r^n\left(h_x,y_j,\theta_k\right)+$$

$$+\frac{1}{2h_y}\left(v^n\left(0,y_j+0.5h_y,\theta_k\right)\overline{c}_r^n\left(0,y_j+h_y,\theta_k\right)-v^n\left(0,y_j-0.5h_y,\theta_k\right)\overline{c}_r^n\left(0,y_j-h_y,\theta_k\right)\right)+\frac{1}{2H\left(0,y_j\right)h_\theta}\cdot$$

$$\cdot\left(w_r'^n\left(0,y_j,\theta_k+0.5h_\theta\right)\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-w_r'^n\left(0,y_j,\theta_k-0.5h_\theta\right)\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)\right)=\frac{1}{h_x^2}\left(\mu_{h,r}\left(0.5h_x,y_j,\theta_k\right)+\right.$$

$$+\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)\Big)\left(\overline{c}_r^n\left(h_x,y_j,\theta_k\right)-\overline{c}_r^n\left(0,y_j,\theta_k\right)\right)-\mu_{h,r}\left(-0.5h_x,y_j,\theta_k\right)\frac{h_x}{3}\vartheta_1+\frac{1}{h_y^2}\Big(\mu_{h,r}\left(0,y_j+0.5h_y,\theta_k\right)\cdot \tag{90}$$

$$\cdot\left(\overline{c}_r^n\left(0,y_j+h_y,\theta_k\right)-\overline{c}_r^n\left(0,y_j,\theta_k\right)\right)-\mu_{h,r}\left(0,y_j-0.5h_y,\theta_k\right)\left(\overline{c}_r^n\left(0,y,\theta\right)-\overline{c}_r^n\left(0,y_j-h_y,\theta_k\right)\right)\Big)+$$

$$+\frac{1}{H^2\left(0,y_j\right)h_\theta^2}\Big(\mu_{\nu,r}\left(0,y_j,\theta_k+0.5h_\theta\right)\left(\overline{c}_r^n\left(0,y_j,\theta_k+h_\theta\right)-\overline{c}_r^n\left(0,y_j,\theta_k\right)\right)-\mu_{\nu,r}\left(0,y_j,\theta_k-0.5h_\theta\right)\cdot$$

$$\cdot\left(\overline{c}_r^n\left(0,y_j,\theta_k\right)-\overline{c}_r^n\left(0,y_j,\theta_k-h_\theta\right)\right)\Big)+\overline{F}_r^n,\ \left(0,y_j,\theta_k\right)\in\overline{\omega}^+,\ r=1,2,3,\ n=1,\ldots,N_T.$$

The error of the approximation of scheme (90) at the boundary nodes of the grid $\overline{\omega}^+$ in case of $x_i = 0$ is equal to $O\left(\tau+h_x^2\right)$.

For the case when the boundary condition (11) is satisfied and $x_i = L_i$, as well as for the cases of boundary conditions (12) and (13), the methods for constructing the difference scheme for the problem (8)–(13) are analogous to those described above, starting from relation (43). Due to the complexity of their description within this article, they are not provided.

Discussion and Conclusion. A second-order difference scheme for approximation on a uniform grid is proposed, which approximates the initial-boundary value problem for the three-dimensional diffusion-convection equation of multifractional suspensions at all nodes of the uniform grid, including boundary nodes. Special attention is given to the description of the approximation methods at the boundary nodes of the grid using an extended grid. The proposed scheme has an approximation error in the norm of the grid space C: second order with respect to the spatial grid steps and first order accuracy with respect to the time step. Further research is focused on proving the stability and convergence of the constructed difference scheme based on the grid maximum principle under mild constraints on the grid Peclet number, the satisfaction of the Courant condition, the aforementioned smoothness conditions, and other restrictions that are naturally satisfied for discrete models of hydro-physical coastal systems.

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Conflict of Interest Statement: **the author declares no conflict of interest.**

The author has read and approved the final version of manuscript.

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Конфликт интересов: **автор заявляет об отсутствии конфликта интересов.**

Автор прочитал и одобрил окончательный вариант рукописи.

Received / Поступила в редакцию 19.08.2024
Reviewed / Поступила после рецензирования 02.09.2024
Accepted / Принята к публикации 02.09.2024