COMPUTATIONAL MATHEMATICS
ВЫЧИСЛИТЕЛЬНАЯ МАТЕМАТИКА

UDC 519.6

Original Theoretical Research

https://doi.org/10.23947/2587-8999-2024-8-4-7-18

Increasing the Accuracy of Solving Boundary Value Problems with Linear Ordinary Differential Equations Using the Bubnov-Galerkin Method

Natalya K. Volosova[1], Konstantin A. Volosov[2], Aleksandra K. Volosova[2], Mikhail I. Karlov[3], Dmitriy F. Pastukhov[4] ✉, Yuriy F. Pastukhov[4]

[1] MGTU named after N.E. Bauman, Moscow, Russian Federation

[2] Russian University of Transport, Moscow, Russian Federation

[3] Moscow Institute of Physics and Technology (National Research University), Dolgoprudny, Russian Federation

[4] Polotsk State University named after Euphrosyne of Polotsk, Novopolotsk, Republic of Belarus

✉ dmitrij.pastuhov@mail.ru

Abstract

Introduction. This study investigates the possibility of increasing the accuracy of numerically solving boundary value problems using the modified Bubnov-Galerkin method with a linear ordinary differential equation, where the coefficients and the right-hand side are continuous functions. The order of the differential equation n must be less than the number of coordinate functions m.

Materials and Methods. A modified Petrov-Galerkin method was used to numerically solve the boundary value problem. It employs a system of linearly independent power-type basis functions on the interval $[-1,1]$, each normalized by the unit Chebyshev norm. The system of linear algebraic equations includes only the linearly independent boundary conditions of the original problem.

Results. For the first time, an integral quadrature formula with a 22nd order error was developed for a uniform grid. This formula is used to calculate the matrix elements and coefficients in the right-hand side of the system of linear algebraic equations, taking into account the scalar product of two functions based on the new quadrature formula. The study proves a theorem on the existence and uniqueness of a solution for boundary value problems with general non-separated conditions, provided that n linearly independent particular solutions of a homogeneous differential equation of order n are known.

Discussion and Conclusion. The hydrodynamic problem in a viscous strong boundary layer with a third-order equation was precisely solved. The analytical solution was compared with its numerical counterpart, and the uniform norm of their difference did not exceed $5\cdot10^{-15}$. The formulas derived using the generalized Bubnov-Galerkin method may be useful for solving boundary value problems with linear ordinary differential equations of the third and higher orders.

Keywords: hydrodynamics, numerical methods, ordinary differential equations, boundary value problems, Galerkin method

For Citation. Volosova N.K., Volosov K.A., Volosova A.K., Karlov M.I., Pastukhov D.F., Pastukhov Yu.F. Increasing the accuracy of solving boundary value problems with a linear differential equation using the Bubnov-Galerkin method. *Computational Mathematics and Information Technologies*. 2024;8(4):7–18. https://doi.org/10.23947/2587-8999-2024-8-4-7-18

Оригинальное теоретическое исследование

Увеличение точности решения краевых задач с линейными обыкновенными дифференциальными уравнениями методом Бубнова-Галеркина

Н.К. Волосова[1], К.А. Волосов[2], А.К. Волосова[2], М.И. Карлов[3], Д.Ф. Пастухов[4]✉, Ю.Ф. Пастухов[4]

[1] Московский государственный технический университет им. Н.Э. Баумана, г. Москва, Российская Федерация

[2] Российский университет транспорта, г. Москва, Российская Федерация

[3] Московский физико-технический институт (национальный исследовательский университет), г. Долгопрудный, Российская Федерация

[4] Полоцкий государственный университет им. Евфросинии Полоцкой, г. Новополоцк, Республика Беларусь

✉ dmitrij.pastuhov@mail.ru

Аннотация

Введение. Исследуется возможность увеличения точности численного решения краевой задачи модифицированным методом Бубнова-Галеркина с линейным обыкновенным дифференциальным уравнением, в котором коэффициенты и правая часть являются непрерывными функциями. Порядок дифференциального уравнения n должен быть меньше числа координатных функций m.

Материалы и методы. Для численного решения краевой задачи использован модифицированный метод Петрова-Галеркина с системой линейно независимых базисных функций степенного вида на отрезке $[-1,1]$ с единичной нормой Чебышева для каждой функции системы. В систему линейных алгебраических уравнений включены только линейно независимые краевые условия исходной задачи.

Результаты исследования. Впервые построена интегральная квадратурная формула на равномерной сетке с двадцать вторым порядком погрешности для вычисления элементов матрицы и коэффициентов правой части системы линейных алгебраических уравнений с учетом скалярного произведения двух функций по новой квадратурной формуле. Доказана теорема существования и единственности решения краевой задачи с неразделенными краевыми условиями общего вида, если известны n линейно независимых частных решений однородного дифференциального уравнения порядка n.

Обсуждение и заключение. Точно решена гидродинамическая задача в вязком сильном пограничном слое с уравнением третьего порядка. Аналитическое решение сравнено с численным решением, равномерная норма разности решений не превышает $5 \cdot 10^{-15}$. Полученные обобщенным методом Бубнова-Галеркина формулы могут быть полезными для решения краевых задач с линейными обыкновенными дифференциальными уравнениями третьего и более высоких порядков.

Ключевые слова: гидродинамика, численные методы, обыкновенные дифференциальные уравнения, краевые задачи, метод Галеркина

Для цитирования. Волосова Н.К., Волосов К.А., Волосова А.К., Карлов М.И., Пастухов Д.Ф., Пастухов Ю.Ф. Увеличение точности решения краевых задач с линейными обыкновенными дифференциальными уравнениями методом Бубнова-Галеркина. *Computational Mathematics and Information Technologies*. 2024;8(4):7–18. https://doi.org/10.23947/2587-8999-2024-8-4-7-18

Introduction. The most well-known methods for solving boundary value problems for ordinary differential equations on an interval are the shooting method [1] and the tridiagonal matrix algorithm [1]. These methods determine the unknown function on a given grid (grid function) using difference equations. In this study, a boundary value problem in the boundary layer of a viscous incompressible fluid, described by a third-order ordinary differential equation [2–3], is considered, and its numerical solution is obtained in a functional form. The hydrodynamic problem in the viscous boundary layer is solved using the modified Bubnov-Galerkin method [4] with a system of linearly independent basis functions. These basis functions have a simple power form, are defined on the interval $[-1,1]$, and are normalized using the unit Chebyshev norm. The unknown solution function is expanded into a series of linearly independent basis functions. In this study, the existence and uniqueness theorem for a boundary value problem on the interval $[a, b]$ with a linear ordinary differential equation of arbitrary order n is generalized to the case of non-separated boundary conditions.

A new integral quadrature formula for a uniform grid, with the number of intervals being a multiple of twenty, is developed for the first time in this work. The quadrature formula achieves a 22nd order error. Compared to the previous work [4], the new quadrature formula, applied to calculate the matrix elements and coefficients in the right-hand side of the system of linear algebraic equations using the scalar product of two functions, reduces the Chebyshev norm of the problem's error by an order of magnitude. For a third-order equation, the system of linear algebraic equations includes $n-1$ linearly independent boundary conditions and $m-n+1$ orthogonality conditions for the residual of the differential equation to the basis functions [4–5] (n is the order of the ODE, and m is the number of basis functions). An exact solution to the problem [1] with selected parameters was obtained, allowing the computation of the Chebyshev norm of the difference between the exact and numerical solutions. Methods for high-accuracy computations in hydrodynamics are presented in works [6–8].

Materials and Methods

Problem Formulation. Let the unknown function u(x) be continuously differentiable n-times on the interval $u(x)\in C^n[a,b]$, be the solution to a boundary value problem governed by an *n*-th order linear ordinary differential equation with variable coefficients $g_i(x)\in C[a,b], i=\overline{0,n}$:

$$\begin{cases} L[u(x)] = f(x),\ x\in(a,b) \\ L[u(x)] \equiv \left(\sum_{i=0}^{n} g_i(x)\dfrac{d^i}{dx^i}\right)u(x), \end{cases} \tag{1}$$

$$\sum_{i=0}^{n-1}\left(\alpha_\mu^i u^{(i)}(a)+\beta_\mu^i u^{(i)}(b)\right)=\gamma_\mu,\ \mu=\overline{1,n}. \tag{2}$$

In the boundary value problem (1)–(2) the functions $g_i(x)(i=\overline{0,n}), f(x)\in C[a,b]$ are given and continuous on the interval [*a*, *b*]. The boundary conditions (2) are specified as linear forms of the function and its derivatives up to the *n*–1-th order at the points $x=a$, $x=b$. These conditions are of a general type. For the problem (1) to be well-posed, the total number of boundary conditions must equal *n*. The coefficient matrices $\alpha_\mu^i,\beta_\mu^i, i=\overline{0,n-1},\mu=\overline{1,n}$, and scalars $\gamma_\mu,\ \mu=\overline{1,n}$ defining the boundary conditions are given. The relationship between these parameters α_μ^i,β_μ^i determines the existence and uniqueness of the solution to the boundary value problem (1)–(2).

Theorem 1. Let *n* linearly independent particular solutions of the homogeneous equation (1) be known $U_j(x), j=\overline{1,n}$. Then the boundary value problem (1)–(2) has a unique solution if and only if the following condition is satisfied: $\det A_{\mu j}\neq 0,\mu=\overline{1,n}, j=\overline{1,n}$, where $A_{\mu j}=\sum_{i=0}^{n-1}\left(\alpha_\mu^i U_j^{(i)}(a)+\beta_\mu^i U_j^{(i)}(b)\right),\mu, j=\overline{1,n}.$

Proof. The general solution of the linear inhomogeneous equation (1) is given by:

$$u(x)=\sum_{j=1}^{n}U_j(x)D_j+\overline{u(x)},\ j=\overline{1,n},$$

where D_j are arbitrary constants of integration, $\overline{u(x)}$ is a particular solution of the inhomogeneous equation (1), and $L[\overline{u(x)}]=f(x)$; $U_j(x)$ are linearly independent particular solutions of the corresponding homogeneous equation $L[U_j(x)]=0, j=1,n$.

Substituting the solution *u*(*x*) into the boundary conditions (2) gives:

$$\begin{gathered}\sum_{i=0}^{n-1}\left(\alpha_\mu^i u^{(i)}(a)+\beta_\mu^i u^{(i)}(b)\right)=\sum_{i=0}^{n-1}\left[\alpha_\mu^i\left(\sum_{j=1}^{n}U_j^{(i)}(a)D_j+\overline{u^{(i)}(a)}\right)+\beta_\mu^i\left(\sum_{j=1}^{n}U_j^{(i)}(b)D_j+\overline{u^{(i)}(b)}\right)\right]=\gamma_\mu \Leftrightarrow \\ \sum_{j=1}^{n}\left(\sum_{i=0}^{n-1}\alpha_\mu^i U_j^{(i)}(a)+\beta_\mu^i U_j^{(i)}(b)\right)D_j=\gamma_\mu-\sum_{i=0}^{n-1}\alpha_\mu^i\overline{u^{(i)}(a)}+\beta_\mu^i\overline{u^{(i)}(b)},\mu=\overline{1,n}.\end{gathered} \tag{3}$$

An inhomogeneous system of *n* linear algebraic equations (3) with respect to *n* unknowns $D_j, j=\overline{1,n}$ has a unique solution if and only if the matrix *A* is non-singular, that is, $\det A_{\mu j}\neq 0,\mu=\overline{1,n}, j=\overline{1,n}$,

$$A_{\mu j}=\sum_{i=0}^{n-1}\left(\alpha_\mu^i U_j^{(i)}(a)+\beta_\mu^i U_j^{(i)}(b)\right),\mu, j=\overline{1,n}. \tag{4}$$

Theorem 1 is proven. It should be noted that Theorem 1 generalizes Statement 1 from the work [4, p. 25] for the case of separated boundary conditions. In the works on hydrodynamics [2–3], T.Ya. Ershova presents a two-dimensional hydrodynamic problem for a viscous layer, taking into account the continuity equation in incompressible fluid and the fluid dynamics equation, which is reduced, using self-similar variables, to a third-order differential equation in the strong boundary layer:

$$\begin{cases} Lu\equiv\varepsilon u'''(x)+ru''(x)=f(x),\ x\in(0,1),\varepsilon\in(0,1),\ r=const>0, \\ u(0)=0,\ u(1)=0,\ u'(1)=0. \end{cases} \tag{5}$$

We will solve problem (5) analytically for a particular case of the right-hand side of the equation $f(x)\equiv 1, \varepsilon u'''(x)+ru''(x)\equiv 1,\ z(x)=u''(x),\ \ \varepsilon z'(x)+rz(x)=1$.

Let us write the general solution of the homogeneous equation $\varepsilon z'(x)+rz(x)=0 \Leftrightarrow z'(x)=-\frac{r}{\varepsilon}z(x) \Leftrightarrow z(x)=C_0\exp\left(-\frac{r}{\varepsilon}x\right)$.

By integrating the last solution twice, we obtain $u'(x)=\int z(x)dx=-C_0\frac{\varepsilon}{r}\exp\left(-\frac{r}{\varepsilon}x\right)+C_1, u(x)=\int u'(x)dx=C_0\frac{\varepsilon^2}{r^2}\exp\left(-\frac{r}{\varepsilon}x\right)+C_1x+C_2.$

A particular solution of the equation $\varepsilon u'''(x)+ru''(x)\equiv 1$ is sought in the form:

$$\overline{u(x)}=Cx^2, \overline{u}'''(x)=0,\ 2rC=1 \Leftrightarrow C=\frac{1}{2r},\ \overline{u(x)}=\frac{x^2}{2r},$$

$$u_{on}(x)=u_{oo}(x)+\overline{u(x)}=C_0\frac{\varepsilon^2}{r^2}\exp\left(-\frac{r}{\varepsilon}x\right)+C_1x+C_2+\frac{x^2}{2r}, u'_{on}(x)=-C_0\frac{\varepsilon}{r}\exp\left(-\frac{r}{\varepsilon}x\right)+C_1+\frac{x}{r}. \quad (6)$$

Function (6) is the general solution of equation (5) with the right-hand side $f(x)\equiv 1$. We apply the boundary conditions of problem (5):

$$u(0)=0 \Leftrightarrow C_0\frac{\varepsilon^2}{r^2}+C_2=0 \Leftrightarrow C_2=-C_0\frac{\varepsilon^2}{r^2}\cdot$$

$$u(1)=0 \Leftrightarrow C_0\frac{\varepsilon^2}{r^2}\exp\left(-\frac{r}{\varepsilon}\right)+C_1+C_2+\frac{1}{2r}=0, u'(1)=0 \Leftrightarrow -C_0\frac{\varepsilon}{r}\exp\left(-\frac{r}{\varepsilon}\right)+C_1+\frac{1}{r}=0.$$

Then

$$C_1=C_0\frac{\varepsilon}{r}\exp\left(-\frac{r}{\varepsilon}\right)-\frac{1}{r},\ \ C_0\frac{\varepsilon^2}{r^2}\exp\left(-\frac{r}{\varepsilon}\right)+C_1+C_2+\frac{1}{2r}=0 \Leftrightarrow$$

$$C_0\frac{\varepsilon^2}{r^2}\exp\left(-\frac{r}{\varepsilon}\right)+C_0\frac{\varepsilon}{r}\exp\left(-\frac{r}{\varepsilon}\right)-\frac{1}{r}-C_0\frac{\varepsilon^2}{r^2}+\frac{1}{2r}=0 \Leftrightarrow C_0=\frac{r}{2\left(\left(\varepsilon^2+\varepsilon r\right)\exp\left(-\frac{r}{\varepsilon}\right)-\varepsilon^2\right)},$$

$$C_2=-C_0\frac{\varepsilon^2}{r^2}=\frac{\varepsilon^2}{2r\left(\varepsilon^2-\left(\varepsilon^2+\varepsilon r\right)\exp\left(-\frac{r}{\varepsilon}\right)\right)}, C_1=\frac{\varepsilon\exp\left(-\frac{r}{\varepsilon}\right)}{2\left(\left(\varepsilon^2+\varepsilon r\right)\exp\left(-\frac{r}{\varepsilon}\right)-\varepsilon^2\right)}-\frac{1}{r}.$$

As a result, we obtain:

$$u(x)=\frac{\varepsilon^2\exp\left(-\frac{r}{\varepsilon}x\right)}{2r\left(\left(\varepsilon^2+\varepsilon r\right)\exp\left(-\frac{r}{\varepsilon}\right)-\varepsilon^2\right)}+\left(\frac{\varepsilon\exp\left(-\frac{r}{\varepsilon}\right)}{2\left(\left(\varepsilon^2+\varepsilon r\right)\exp\left(-\frac{r}{\varepsilon}\right)-\varepsilon^2\right)}-\frac{1}{r}\right)x+\frac{\varepsilon^2}{2r\left(\varepsilon^2-\left(\varepsilon^2+\varepsilon r\right)\exp\left(-\frac{r}{\varepsilon}\right)\right)}+\frac{x^2}{2r}. \quad (7)$$

For testing the program using the Bubnov-Galerkin algorithm [2–4], we choose the parameters $r = 1$, $\varepsilon = 1/2$, and from formula (7) we obtain the function (8).

$$u(x)=\frac{\exp(2-2x)}{2\left(3-e^2\right)}+\frac{\left(e^2-2\right)x}{\left(3-e^2\right)}-\frac{e^2}{2\left(3-e^2\right)}+\frac{x^2}{2}. \quad (8)$$

As in the work [4], we choose a system of basis functions $\phi_i(x), i=\overline{0,m},\ m>n,$ that is linearly independent.

$$\{\phi_i(x)\}_{i=0}^m=\left\{\left(\frac{2x-a-b}{b-a}\right)^i, x\in[a,b], i=\overline{0,m}\right\}, \|\phi_i(x)\|_C=\max_{x\in[a,b]}|\phi_i(x)|=1\forall i=\overline{0,m}. \quad (9)$$

A linear transformation $z=\frac{2x-a-b}{b-a}, z\in[-1,1]$ bijectively maps the interval $[a, b]$ to the interval $[-1,1]$. The basis functions $\phi_i(x)$ with even indices are even on the interval $[-1,1]$, while those with odd indices are odd. Let the midpoint of the interval $[a, b]$ be denoted by $c = (a + b)/2$. We expand the solution of the general problem (1) in terms of the system of basis functions (9):

$$u(x)=u(c)+\sum_{j=1}^{m}\phi_j(x)D_j=u(c)+\sum_{j=1}^{m}\left(\frac{2(x-c)}{b-a}\right)^j D_j\,. \quad (10)$$

From formula (10), the identity follows $u(c) = u(c)$, and formula (10) itself is the expansion of the unknown function in a power series centered at $x = c = (a + b)/2$.

Note. The Bubnov-Galerkin method is orthogonal. However, it does not require the system of basis functions to be orthogonal polynomials, such as Legendre polynomials on the interval [–1,1] with a weight function $\rho(z)\equiv 1$, $z\in[-1,1]$. The system of functions (9) must be linearly independent. It should be noted that the orthogonality of the system of functions (9) in the residual of the differential equation in problem (1) is expressed with the weight function $\rho(z)\equiv 1$, $z\in[-1,1]$. Moreover, finding derivatives of any order from functions (9) is significantly easier than from Legendre polynomials.

Substitute (10) into equation (1) and write the residual of equation (1):

$$R(u((x))=L[u(x)]-f(x)=L\left(u(c)+\sum_{j=1}^{m}\phi_j(x)D_j\right)-f(x)=L\big(u(c)\big)+\sum_{j=1}^{m}L\phi_j(x)D_j-f(x).$$

According to the Bubnov-Galerkin method, we write the orthogonality conditions of the residual with respect to the maximum number of coordinate functions $\{1,z,z^2,...,z^{m-3}\}$, for solving problem (5) with the third-order equation, which contributes the most to the residual of equation (5) (for m–n+1 functions in the general problem (1)):

$$\big\langle R\big(u(x)\big),\phi_i(x)\big\rangle=0,\overline{i=0,m-3}\Leftrightarrow\sum_{j=1}^{n}\big\langle L\phi_j(x),\phi_i(x)\big\rangle D_j=\big\langle f(x)-L\big(u(c)\big),\phi_i(x)\big\rangle, i=\overline{0,m-3}, \tag{11}$$

In formula (11), the symbol $\langle q, g\rangle$ denotes the scalar product of functions:

$$\langle q,g\rangle=\int_a^b q(x)g(x)dx,\ \ L\big(u(c)\big)=g_0(x)u(c)=g_0(x)u_c. \tag{12}$$

To ensure the closure of the system (11), two additional equations are required. As shown in work [9], the boundary value problem requires that the solution belongs to the class of admissible functions, meaning that all linearly independent boundary conditions should be used. At the endpoints of the interval $x = a$, $x = b$, using formula (10), we obtain:

$$u(a)\equiv u_a=u(c)+\sum_{j=1}^{m}\left(\frac{(2a-a-b}{b-a}\right)^j D_j=u_c+\sum_{j=1}^{m}(-1)^j D_j, u(b)\equiv u_b=u(c)+\sum_{j=1}^{m}\left(\frac{(2b-a-b}{b-a}\right)^j D_j=u_c+\sum_{j=1}^{m}D_j.$$

By summing the last two equations and expressing $u(c) = u_c$, we get

$$u_c=\left(\frac{u_a+u_b}{2}\right)-D_2-D_4-...-\begin{cases}D_m, m=2l\\ D_{m-1}, m=2l+1.\end{cases} \tag{13}$$

Similarly, expressing $\frac{u_b-u_a}{2}$, we obtain the formula:

$$\frac{u_b-u_a}{2}=D_1+D_3+...+\begin{cases}D_{m-1}, m=2l\\ D_m, m=2l+1.\end{cases} \tag{14}$$

We compute the first derivative $u(x)$ from formula (10) and set it to zero at the point $x = b$ according to the boundary condition of problem (5):

$$u'(x)=\sum_{j=1}^{m}\phi'_j(x)D_j=\sum_{j=1}^{m}\frac{2j}{(b-a)}\left(\frac{(2x-a-b}{b-a}\right)^{j-1}_{x=b}D_j=0\Leftrightarrow D_1+2D_2+3D_3+...+mD_m=0. \tag{15}$$

Substitute the value of $u(c)$ from formula (13) into the right-hand side of equation (11), then move all terms containing D_j to the left-hand side of equation (11). Taking into account formulas (14) and (15), we obtain a system of linear algebraic equations (16) for the unknown coefficients D_j

$$\sum_{j=1}^{m}a_{i,j}D_j=\overline{f_i},\ i=\overline{0,m-1}, \tag{16}$$

where the elements of the matrix $a_{i,j}, i=\overline{0,m-1}, j=\overline{1,m}$ and the coefficients of the right-hand side $\overline{f_i}$ of system (16) are given as:

$$a_{i,j}=\begin{cases}\big\langle L\phi_j,\phi_i\big\rangle,\ \text{if}\, j\equiv 1\big(\text{mod}\,2\big), i=\overline{0,m-3}\\ \big\langle L\big(\phi_j-1\big),\phi_i\big\rangle,\ \text{if}\, j\equiv 0\big(\text{mod}\,2\big), i=\overline{0,m-3}\\ 1,\ \text{if}\ i=m-2, j\equiv 1\big(\text{mod}\,2\big)\\ 0,\ \text{if}\ i=m-2, j\equiv 0\big(\text{mod}\,2\big)\\ j,\ \text{if}\ i=m-1\end{cases},$$

$$\overline{f_i} = \begin{cases} \left\langle f(x) - L\left(\frac{u_a+u_b}{2}\right), \phi_i(x) \right\rangle, \text{ if } i = \overline{0, m-3}, L\left(\frac{u_a+u_b}{2}\right) = \left(\frac{u_a+u_b}{2}\right) g_0(x) \\ \frac{u_b - u_a}{2}, \text{ if } i = m-2 \\ 0, \text{ if } i = m-1 \end{cases},$$

The action of a third-order linear differential operator in problem (5) on a basis function with index *j* can be represented by the system of formulas (17):

$$L\phi_j = \begin{cases} g_0(x), \text{ if } j = 0, \\ \frac{2g_1(x)}{(b-a)} + g_0(x)\left(\frac{2x-a-b}{b-a}\right), \text{ if } j = 1, \\ 8g_2(x)\frac{1}{(b-a)^2} + +4g_1(x)\frac{(2x-a-b)}{(b-a)^2} + g_0(x)\left(\frac{2x-a-b}{b-a}\right)^2, \text{ if } j = 2, \\ 8j(j-1)(j-2)g_2(x)\frac{(2x-a-b)^{j-3}}{(b-a)^j} + 4j(j-1)g_2(x)\frac{(2x-a-b)^{j-2}}{(b-a)^j} + \\ +2jg_1(x)\frac{(2x-a-b)^{j-1}}{(b-a)^j} + g_0(x)\frac{(2x-a-b)^j}{(b-a)^j}, \text{ if } j \geq 3. \end{cases}$$

The numerical solution of problem (5) is obtained by substituting (13) into formula (10), resulting in the expression (18):

$$u(x) = \left(\frac{u_a+u_b}{2}\right) + \sum_{j=1}^{m}\left[\left(\frac{(2x-a-b}{b-a}\right)^j + \left(\frac{-1+(-1)^{j+1}}{2}\right)\right] D_j. \quad (18)$$

In solution (18), the unknown vector D is determined from the system of linear algebraic equations (16) $D = A^{-1}\overline{f}.$. The estimate of the uniform norm of the solution $u(x)$ yields the following result:

$$|u(x)| \leq \frac{|u_a|+|u_b|}{2} + 2\sum_{j=1}^{m}|D_j| \leq \frac{|u_a|+|u_b|}{2} + 2m\max_{j=1,m} D_j = \frac{|u_a|+|u_b|}{2} + 2m\|D\|_C \leq \frac{|u_a|+|u_b|}{2} + 2m\|A^{-1}\|_C\|\overline{f}\|_C \Rightarrow$$

$$\|u\|_C \leq \frac{|u_a|+|u_b|}{2} + 2m\|A^{-1}\|_C\|\overline{f}\|_C.$$

where the norm of the inverse matrix A^{-1} is determined by the formula $\|A^{-1}\|_C = \max_{i=1,m}\sum_{j=1}^{m}|a_{i,j}^{-1}|.$

In work [4], to compute all matrix elements $a_{i,j}$, and the coefficients of the right-hand side $\overline{f_i}$ of the system of linear algebraic equations (16) through the scalar product of two functions (12), a composite quadrature integral formula (19) with a uniform step and 12th-order error $O(h^{12})$ was applied. This formula was used by the program for the numerical solution of test example 3:

$$\langle y_1, y_2 \rangle = \int_a^b y_1(x)y_2(x)dx = 5h\sum_{i=0}^{n_1} y_1(x_i)\, y_2(x_i) C_i + O(h^{12}), n_1 = 10p, h = \frac{b-a}{n_1}, p \in N, \quad (19)$$

where the weight coefficients of the integral quadrature formula (19) are determined by the remainder of the division of the node index *i* on the uniform grid by 10.

$$C_i = \begin{cases} \frac{16067}{299376}, i = 0 \vee i = n_1, \\ \frac{16067}{149688}, (i \equiv 0 \bmod 10) \wedge (0 < i < n_1), \\ \frac{26575}{74844}, (i \equiv 1 \bmod 10) \vee (i \equiv 9 \bmod 10), \\ \frac{-16175}{99792}, (i \equiv 2 \bmod 10) \vee (i \equiv 8 \bmod 10), \\ \frac{5675}{6237}, (i \equiv 3 \bmod 10) \vee (i \equiv 7 \bmod 10), \\ \frac{-4825}{5544}, (i \equiv 4 \bmod 10) \vee (i \equiv 6 \bmod 10), \\ \frac{17807}{12474}, i \equiv 5 \bmod 10. \end{cases} \quad (20)$$

Applying the scalar product formula for two functions using formula (19) with weight coefficients (20) and the Petrov-Galerkin algorithm (16)–(18) with $m = 16$ basis functions and $n_1 = 20$ integration intervals, the program computes the uniform norm of the residual in problem (5). The exact solution is calculated using formula (8) on a uniform grid:

$$u_i^{exact}, x_i = a + h \cdot i, i = \overline{0, n_1}, h = \frac{b-a}{n_1},$$

$$\left\| u^{num} - u^{exact} \right\|_C = \max_{i=0,n_1} \left| u_i^{num} - u_i^{exact} \right| \approx 2.016997679987753\text{E}{-}012.$$

In this study, a novel quadrature integral formula with a uniform step and a 22nd-order error $O(h^{22})$ is proposed for the first time. This formula enhances the accuracy of the scalar product computation and the evaluation of matrix elements and the coefficients of the right-hand side of the system of linear algebraic equations (16), thereby reducing the norm of the residual error in problem (5).

For the quadrature integral formula on a uniform grid $[a, b]$ [1, p. 87], with an error of $O(h^{22})$ and considering symmetry, the quadrature formula is written relative to the midpoint of the interval $[a, b]$:

$$c = (a+b)/2 \Leftrightarrow z = 0, \quad x = \frac{(a+b)}{2} + \frac{(b-a)}{2} z, z \in [-1,1], x \in [a,b], dx = \frac{(b-a)}{2} dz$$

we obtain:

$$\int_a^b f(x)dx = \frac{(b-a)}{2} \int_{-1}^{1} f(z)dz, \int_{-1}^{1} f(z)dz = C_0 f(0) + \sum_{i=1}^{i=10} C_i \left(f(-z_i) + f(z_i) \right), z_i = i/10. \tag{21}$$

By substituting even-degree power functions into formula (21), we simplify the computations as follows (odd-degree functions result in the trivial identity 0=0) $f(z) = \{0, z^2, z^4, z^6, z^8, z^{10}, z^{12}, z^{14}, z^{16}, z^{18}, z^{20}\}$, we obtain a system of 11 linear inhomogeneous algebraic equations for the variables $C_i, i = \overline{0,10}$:

$$I_{2k} = \int_{-1}^{1} z^{2k} dz = \frac{2}{2k+1} = \begin{cases} C_0 + 2\sum_{i=1}^{10} C_i, k = 0 \\ 2\sum_{i=1}^{10} z_i^{2k} C_i, k = 1,2,3,4,5,6,7,8,9,10; z_i = i/10, i = \overline{1,10}. \end{cases} \tag{22}$$

Results. The system of equations (22) was solved symbolically without rounding using a symbolic computation environment. The coefficients C_i are provided in formula (24), and Table 1 illustrates the comparison between computed and exact values. The table presents the numerical values of $I^{num}{}_k$ (the right-hand side of formula (22)) and the exact values of the integral $I^{exact}{}_k = \frac{1+(-1)^k}{k+1}$ (the left-hand side of formula (22)) for power functions $f(z) = \{1, z, z^2, \ldots z^{21}, z^{22}\}$ on the interval [–1, 1] taking into account the coefficients.

Table 1

Comparison of Numerical and Exact Values

k	$I^{exact}{}_k = \frac{1+(-1)^k}{k+1}$	$I^{num}{}_k$
0	2.00000000000000	2.00000000000003
1	0.000000000000000E+000	1.379105163401562E–015
2	0.666666666666667	0.666666666666679
3	0.000000000000000E+000	2.586472702681419E–015
4	0.400000000000000	0.399999999999999
5	0.000000000000000E+000	–3.694961003830599E–016
6	0.285714285714286	0.285714285714285
7	0.000000000000000E+000	–4.388850394221322E–016
8	0.222222222222222	0.222222222222222
9	0.000000000000000E+000	–1.752070710736575E–016
10	0.181818181818182	0.181818181818181
11	0.000000000000000E+000	–1.613292832658431E–016
12	0.153846153846154	0.153846153846153

End of table 1

k	$I^{exact}{}_k = \dfrac{1+(-1)^k}{k+1}$	$I^{num}{}_k$
13	0.000000000000000E+000	–8.500145032286355E–017
14	0.133333333333333	0.133333333333333
15	0.000000000000000E+000	2.602085213965211E–017
16	0.117647058823529	0.117647058823529
17	0.000000000000000E+000	1.092875789865388E–016
18	0.105263157894737	0.105263157894737
19	0.000000000000000E+000	1.578598363138894E–016
20	9.523809523809523E–002	9.523809523809518E–002
21	0.000000000000000E+000	1.717376241217039E–016
22	8.695652173913043E–002	8.695652174444995E–002

The conditions of the system (22) and Table 1 demonstrate that the quadrature formulas (22) and (23) achieve twenty-second order accuracy.

If

$$b-a=nh, n=20s, s\in N, \left(\frac{b-a}{2}\right)=10sh, x_i=a+i\cdot h,\ i=\overline{0,n},$$

then

$$\int_a^b f(x)dx = 10h\sum_{i=0}^{n=20s} C_i f\left(x_i\right)+O\left(h^{22}\right), x_i=a+i\cdot h,\ i=\overline{0,n},$$

The scalar product of two functions, as defined by formulas (19) and (20), is expressed as follows:

$$\left\langle y_1,y_2\right\rangle=\int_a^b y_1(x)y_2(x)dx=10h\sum_{i=0}^{n_1} y_1\left(x_i\right)y_2\left(x_i\right)C_i+O\left(h^{22}\right), n_1=20s, h=\frac{b-a}{n_1}, s\in N, \tag{23}$$

where C_i are the weight coefficients in the composite quadrature integral formula (23), which are obtained by solving the system of linear algebraic equations (22):

$$C_i=\begin{cases}\dfrac{1145302367137}{48426042384720}, \text{ if } i=0 \text{ or } i=n_1,\\ \dfrac{1145302367137}{24213021192360}, \text{ if } \left(i\equiv 0 \bmod 20\right) \text{ and } \left(0<i<n_1\right),\\ \dfrac{335582304250}{1470076286679}, \text{ if} \left(i\equiv 1 \bmod 20\right) \text{ or } \left(i\equiv 19 \bmod 20\right),\\ \dfrac{-19467909708875}{41162136027012}, \text{ if } \left(i\equiv 2 \bmod 20\right) \text{ or } \left(i\equiv 18 \bmod 20\right),\\ \dfrac{8274871497250}{3430178002251}, \text{ if } \left(i\equiv 3 \bmod 20\right) \text{ or } \left(i\equiv 17 \bmod 20\right),\\ \dfrac{-413929922392625}{54882848036016}, \text{ if } \left(i\equiv 4 \bmod 20\right) \text{ or } \left(i\equiv 16 \bmod 20\right),\\ \dfrac{50652939811064}{2450127144465}, \text{ if } \left(i\equiv 5 \bmod 20\right) \text{ or } \left(i\equiv 15 \bmod 20\right),\\ -\dfrac{155790561130375}{3430178002251}, \text{ if } \left(i\equiv 6 \bmod 20\right) \text{ or } \left(i\equiv 14 \bmod 20\right)\\ \dfrac{286955364893000}{3430178002251}, \text{ if } \left(i\equiv 7 \bmod 20\right) \text{ or } \left(i\equiv 13 \bmod 20\right)\\ -\dfrac{502376261017625}{3920203431144}, \text{ if } \left(i\equiv 8 \bmod 20\right) \text{ or } \left(i\equiv 12 \bmod 20\right)\\ \dfrac{1704056522480500}{10290534006753}, \text{ if } \left(i\equiv 9 \bmod 20\right) \text{ or } \left(i\equiv 11 \bmod 20\right)\\ -\dfrac{1684005984173647}{9355030915230}, \text{ if } i\equiv 10 \bmod 20.\end{cases} \tag{24}$$

We will find the characteristic equation and the particular solutions of the homogeneous equation (5) with the chosen parameters:

$$\varepsilon = \frac{1}{2}, r = 1, \varepsilon u'''(x) + r u''(x) = \frac{1}{2} u'''(x) + u''(x) = 0 \Rightarrow \lambda^3 + 2\lambda^2 = 0 \Leftrightarrow \lambda_1 = \lambda_2 = 0, \lambda_3 = -2,$$

These correspond to three linearly independent solutions:

$$\left\{U_1(x) = 1, U_2(x) = x, U_3(x) = \exp(-2x)\right\}, \left\{U_1'(x) = 0, U_2'(x) = 1, U_3'(x) = -2\exp(-2x)\right\},$$

$$\left\{U_1''(x) = 0, U_2''(x) = 0, U_3''(x) = 4\exp(-2x)\right\}.$$

Next, we verify the existence and uniqueness of the solution to the boundary value problem (5) with parameters $\varepsilon = \frac{1}{2}, r = 1$.

Finally, we compute the elements of the matrix, as defined in formula (4), for solving the system of equations:

$$A_{\mu j} = \sum_{i=0}^{n-1} \left(\alpha_\mu^i U_j^{(i)}(a) + \beta_\mu^i U_j^{(i)}(b) \right), \mu, j = \overline{1,n}, \; u(0) = 0, \; u(1) = 0, \; u'(1) = 0 \Leftrightarrow$$

$$\alpha_1^0 = 1; \alpha_1^1 = 0; \alpha_1^2 = 0; \beta_1^0 = 0; \beta_1^1 = 0; \beta_1^2 = 0,$$

$$\alpha_2^0 = 0; \alpha_2^1 = 0; \alpha_2^2 = 0; \beta_2^0 = 1; \beta_2^1 = 0; \beta_2^2 = 0,$$

$$\alpha_3^0 = 0; \alpha_3^1 = 0; \alpha_3^2 = 0; \beta_3^0 = 0; \beta_3^1 = 1; \beta_3^2 = 0,$$

$$A_{11} = \alpha_1^0 U_1^{(0)}(0) + \beta_1^0 U_1^{(0)}(1) + \alpha_1^1 U_1^{(1)}(0) + \beta_1^1 U_1^{(1)}(1) + \alpha_1^2 U_1^{(2)}(0) + \beta_1^2 U_1^{(2)}(1) = 1\cdot 1 + 0\cdot 1 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 = 1,$$

$$A_{21} = \alpha_2^0 U_1^{(0)}(0) + \beta_2^0 U_1^{(0)}(1) + \alpha_2^1 U_1^{(1)}(0) + \beta_2^1 U_1^{(1)}(1) + \alpha_2^2 U_1^{(2)}(0) + \beta_2^2 U_1^{(2)}(1) = 0\cdot 1 + 1\cdot 1 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 + 0\cdot 0 = 1,$$

$$A_{31} = \alpha_3^0 U_1^{(0)}(0) + \beta_3^0 U_1^{(0)}(1) + \alpha_3^1 U_1^{(1)}(0) + \beta_3^1 U_1^{(1)}(1) + \alpha_3^2 U_1^{(2)}(0) + \beta_3^2 U_1^{(2)}(1) = 0\cdot 1 + 0\cdot 1 + 0\cdot 0 + 1\cdot 0 + 0\cdot 0 + 0\cdot 0 = 0,$$

$$A_{12} = \alpha_1^0 U_2^{(0)}(0) + \beta_1^0 U_2^{(0)}(1) + \alpha_1^1 U_2^{(1)}(0) + \beta_1^1 U_2^{(1)}(1) + \alpha_1^2 U_2^{(2)}(0) + \beta_1^2 U_2^{(2)}(1) = 1\cdot 0 + 0\cdot 1 + 0\cdot 1 + 0\cdot 1 + 0\cdot 0 + 0\cdot 0 = 0,$$

$$A_{22} = \alpha_2^0 U_2^{(0)}(0) + \beta_2^0 U_2^{(0)}(1) + \alpha_2^1 U_2^{(1)}(0) + \beta_2^1 U_2^{(1)}(1) + \alpha_2^2 U_2^{(2)}(0) + \beta_2^2 U_2^{(2)}(1) = 0\cdot 0 + 1\cdot 1 + 0\cdot 1 + 0\cdot 1 + 0\cdot 0 + 0\cdot 0 = 1,$$

$$A_{32} = \alpha_3^0 U_2^{(0)}(0) + \beta_3^0 U_2^{(0)}(1) + \alpha_3^1 U_2^{(1)}(0) + \beta_3^1 U_2^{(1)}(1) + \alpha_3^2 U_2^{(2)}(0) + \beta_3^2 U_2^{(2)}(1) = 0\cdot 0 + 0\cdot 1 + 0\cdot 1 + 1\cdot 1 + 0\cdot 0 + 0\cdot 0 = 1,$$

$$A_{13} = \alpha_1^0 U_3^{(0)}(0) + \beta_1^0 U_3^{(0)}(1) + \alpha_1^1 U_3^{(1)}(0) + \beta_1^1 U_3^{(1)}(1) + \alpha_1^2 U_3^{(2)}(0) + \beta_1^2 U_3^{(2)}(1) = 1\cdot 1 + 0\cdot e^{-2} + 0\cdot(-2) +$$
$$+ 0\cdot\left(-2e^{-2}\right) + 0\cdot 4 + 0\cdot 4e^{-2} = 1,$$

$$A_{23} = \alpha_2^0 U_3^{(0)}(0) + \beta_2^0 U_3^{(0)}(1) + \alpha_2^1 U_3^{(1)}(0) + \beta_2^1 U_3^{(1)}(1) + \alpha_2^2 U_3^{(2)}(0) + \beta_2^2 U_3^{(2)}(1) = 0\cdot 1 + 1\cdot e^{-2} + 0\cdot(-2) +$$
$$+ 0\cdot\left(-2e^{-2}\right) + 0\cdot 4 + 0\cdot 4e^{-2} = e^{-2},$$

$$A_{33} = \alpha_3^0 U_3^{(0)}(0) + \beta_3^0 U_3^{(0)}(1) + \alpha_3^1 U_3^{(1)}(0) + \beta_3^1 U_3^{(1)}(1) + \alpha_3^2 U_3^{(2)}(0) + \beta_3^2 U_3^{(2)}(1) = 0\cdot 1 + 0\cdot e^{-2} + 0\cdot(-2) +$$
$$+ 1\cdot(-2e^{-2}) + 0\cdot 4 + 0\cdot 4e^{-2} = -2e^{-2}.$$

Since $\begin{vmatrix} 1 & 0 & 1 \\ 1 & 1 & e^{-2} \\ 0 & 1 & -2e^{-2} \end{vmatrix} = \begin{vmatrix} 1 & e^{-2} \\ 1 & -2e^{-2} \end{vmatrix} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -3e^{-2} + 1 \neq 0,$ according to Theorem 1, the boundary value problem (5) with parameters $\varepsilon = \frac{1}{2}, r = 1$ has a unique solution. The exact solution with the right-hand side $f(x) \equiv 1, \varepsilon = \frac{1}{2}, r = 1$ is given by function (8). No other solutions exist.

The inverse matrix A^{-1} in the system of linear algebraic equations (SLAE) (16) is computed using the msimsl linear algebra library to find the coefficient vector D_j, $j = 1, m$. The program, using formulas (16), (17), (18), (23), and (24), provides the numerical solution to problem (5)

$$x_i = a + h\cdot i, i = \overline{0, n_1}, h = \frac{b-a}{n_1}, n_1 = 20, a = 0, b = 1.$$

with parameters $\varepsilon = \frac{1}{2}, r = 1, f(x) \equiv 1, g_3(x) = \varepsilon = \frac{1}{2}, g_2(x) = r = 1, g_1(x) = 0, g_0(x) = 0,$ as presented in Table 2. The number of basis functions is $m = 18$.

Table 2

Numerical u_i^{num} and Exact u_i^{exact} Solutions to Problem (5)

x_i	u_i^{num}	u_i^{exact}	$u_i^{num} - u_i^{exact}$
0.000000E+000	0.0000000000E+000	0.00000000000E+000	0.00000000E+000
5.0000000E-002	1.99620012886537E–002	1.99620012886524E–002	1.231653667E–015
0.1000000000	3.48011017443269E–002	3.48011017443259E–002	1.006139616E–015
0.1500000000	4.52427162923431E–002	4.52427162923431E–002	6.938893903E–018
0.2000000000	5.19432275007362E–002	5.19432275007366E–002	–3.816391647E–016
0.2500000000	5.54965548776091E–002	5.54965548776098E–002	–6.800116025E–016
0.3000000000	5.64400990171948E–002	5.64400990171955E–002	–7.563394355E–016
0.3500000000	5.52601200856299E–002	5.52601200856307E–002	–8.118505867E–016
0.4000000000	5.23966044761353E–002	5.23966044761364E–002	–1.033895191E–015
0.4500000000	4.82476683407243E–002	4.82476683407254E–002	–1.075528555E–015
0.5000000000	4.31735420704626E–002	4.31735420704638E–002	–1.221245327E–015
0.5500000000	3.75001756023005E–002	3.75001756023019E–002	–1.415534356E–015
0.6000000000	3.15225006355969E–002	3.15225006355983E–002	–1.450228825E–015
0.6500000000	2.55073824076967E–002	2.55073824076980E–002	–1.356553758E–015
0.7000000000	1.96962905709227E–002	1.96962905709242E–002	–1.467576060E–015
0.7500000000	1.43077159020167E–002	1.43077159020178E–002	–1.103284130E–015
0.8000000000	9.53935703127063E–003	9.53935703127128E–003	–6.574601973E–016
0.8500000000	5.57009907686709E–003	5.57009907686645E–003	6.435824095E–016
0.9000000000	2.56180398726754E–003	2.56180398726446E–003	3.080001531E–015
0.9500000000	6.609305099950E–004	6.609305099896E–004	5.435105490E–015
1.0000000000	0.0000000000E+000	–1.110223024625157E–016	1.110223024E–016

In the first column of Table 1, the values of the nodes x_i on the uniform grid are given. In the second column, the numerical solution u_i^{num} is recorded, and in the third column, the exact solution u_i^{exact} at the nodes is presented. The last column contains the difference between the numerical and exact solutions $u_i^{num} - u_i^{exact}$.

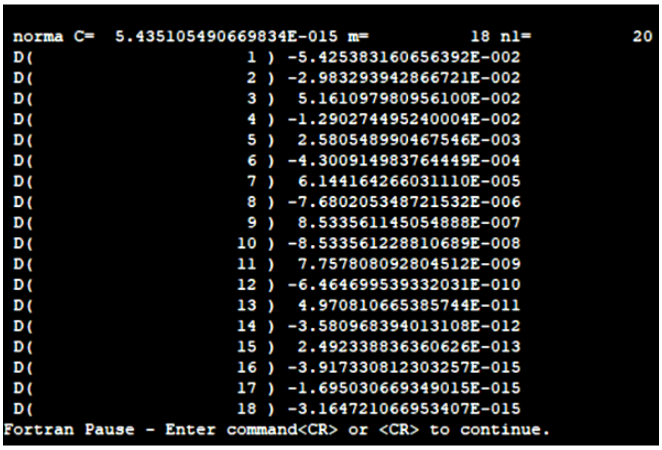

Fig. 1. Program Results

Taking into account formulas (23)–(24), the program gives an error norm $\left\| u^{num} - u^{exact} \right\|_C = \max\limits_{i=0,n_1} \left| u_i^{num} - u_i^{exact} \right| \approx 5.435105\text{E–}015$ with a result several times smaller than when using the scalar product formulas (19)–(20). Fig. 1 shows that the number of basis functions is optimal when the coefficients decrease in absolute value as their index increases. The advantage of the scalar product formulas (23)–(24) over formulas (19)–(20) also lies in the weak dependence of the error norm on the number of basis functions over a wide range of their values.

Discussion and Conclusion. The theorem of existence and uniqueness of the solution to the boundary value problem with a linear ordinary differential equation of order *n* has been generalized to the case of non-separated boundary conditions, provided that *n* linearly independent particular solutions of the corresponding homogeneous equation are

known. The boundary value problem with a third-order differential equation in the boundary layer for an incompressible fluid, with parameters ε = 0,5, $r = 1$ and constant right-hand side $f(x) = 1$ has been solved analytically. A Bubnov-Galerkin method with a system of linearly independent basis functions on the interval [–1,1] has been proposed for the numerical solution of the boundary hydrodynamic problem with a strong boundary layer. The basis functions are bounded by the Chebyshev unit norm. A new quadrature integral formula with a uniform step has been introduced for calculating the matrix elements and right-hand side coefficients in the Bubnov-Galerkin method, which results in a second-order error bound. The Chebyshev vector norm for the difference between the exact solution and the numerical solution on a uniform grid, using the scalar product formulas (23) and (24), is comparable to 10^{-15} and is several orders of magnitude smaller than the norm of the residual using the scalar product formulas (19) and (20) in the same hydrodynamic problem.

References

1. Bahvalov N.S., Zhidkov N.P., Kobelkov G.M. *Numerical methods: a textbook for students of physics and mathematics specialties of higher educational institutions*. Binom. lab. Knowledge; 2011. 636 p. (In Russ.)

2. Ershova, T.Ya. Boundary value problem for a third-order differential equation with a strong boundary layer. *Bulletin of Moscow University. Episode 15: Computational mathematics and cybernetics*. 2020;1:30–39. (In Russ.) https://doi.org/10.3103/S0278641920010057

3. Ershova, T.Ya. On the convergence of a grid solution to the problem for a third-order equation in the case of a strong boundary layer. *Lomonosov readings: scientific conference*. Moscow: MAKS Press LLC, 2020. P. 77–78.

4. Volosova N.K. [et al.]. Modified Bubnov-Galerkin method for solving boundary value problems with a linear ordinary differential equation. *Computational Mathematics and Information Technologies*. 2024;8(3):23–33. (In Russ.) https://doi.org/10.23947/2587-8999-2024-8-3-23-33

5. Bahvalov N.S., Lapin A.V., Chizhonkov E.V. *Numerical methods in problems and exercises*. Moscow: BINOM. Knowledge laboratory; 2010. 240 p. (In Russ.)

6. Petrov A.G. High-precision numerical schemes for solving plane boundary value problems for a polyharmonic equation and their application to problems of hydrodynamics. *Applied Mathematics and Mechanics.* 2023;87(3):343–368 (In Russ.) https://doi.org/10.31857/S0032823523030128

7. Proskurin D.K., Sysoev D.V., Sazonova S.A. Convergence of the computational process when implementing the variational method for solving the boundary value problem of hydrodynamics. *Bulletin of the Voronezh State Technical University.* 2021;17(3):14–19. (In Russ) https://doi.org/10.36622/VSTU.2021.17.3.002

8. Sidoryakina V.V., Solomaha D.A. Symmetrized versions of the Seidel and upper relaxation methods for solving two-dimensional difference problems of elliptic. *Computational Mathematics and Information Technologies*. 2023;7(3):12–19. (In Russ.) https://doi.org/10.23947/2587-8999-2023-7-3-12-19

9. Alekseev V.M., Galeev E.M., Tikhomirov V.M. *Collection of optimization problems: Theory. Examples. Problems*. Moscow: Fizmatlit; 2008. 256 p. (In Russ.)

About the Authors:

Natalya K. Volosova, Post-graduate Student of Bauman Moscow State Technical University (2nd Baumanskaya St. 5–1, Moscow, 105005, Russian Federation), ORCID, navalosova@yandex.ru

Konstantin A. Volosov, Doctor of Physical and Mathematical Sciences, Professor of the Department of Applied Mathematics of the Russian University of Transport (Obraztsova St. 9–9, Moscow, GSP-4, 127994, Russian Federation), ORCID, konstantinvolosov@yandex.ru

Aleksandra K. Volosova, Candidate of Physical and Mathematical Sciences, Chief Analytical Department "Tramplin" LLC, Russian University of Transport (Obraztsova St. 9–9, Moscow, GSP-4, 127994, Russian Federation), ORCID, alya01@yandex.ru

Mikhail I. Karlov, Candidate of Physical and Mathematical Sciences, Associate Professor of the Department of Mathematics, Moscow Institute of Physics and Technology (9, Institutsky Lane, GSP-4, Dolgoprudny, 141701, Russian Federation), ORCID, karlov.mipt@gmail.com

Dmitriy F. Pastukhov, Candidate of Physical and Mathematical Sciences, Associate Professor of Polotsk State University (Blokhin St. 29, Novopolotsk, 211440, Republic of Belarus), ORCID, dmitrij.pastuhov@mail.ru

Yuriy F. Pastukhov, Candidate of Physical and Mathematical Sciences,Associate Professor of Polotsk State University (Blokhin St. 29, Novopolotsk, 211440, Republic of Belarus), ORCID, pulsar1900@mail.ru

***Contributions of the authors:* the authors contributed equally to this article.**

***Conflict of Interest Statement:* the authors declare no conflict of interest.**

All authors have read and approved the final manuscript.

Об авторах:

Наталья Константиновна Волосова, аспирант Московского государственного технического университета им. Н.Э. Баумана (105005, Российская Федерация, г. Москва, ул. 2-я Бауманская, 5, стр. 1), ORCID, navalosova@yandex.ru

Константин Александрович Волосов, доктор физико-математических наук, профессор кафедры прикладной математики Российского университета транспорта (127994, ГСП-4, Российская Федерация, г. Москва, ул. Образцова, 9, стр. 9), ORCID, konstantinvolosov@yandex.ru

Александра Константиновна Волосова, кандидат физико-математических наук, начальник аналитического отдела ООО «Трамплин» Российского университета транспорта (127994, ГСП-4, Российская Федерация, г. Москва, ул. Образцова, 9, стр. 9), ORCID, alya01@yandex.ru

Михаил Иванович Карлов, кандидат физико-математических наук, доцент кафедры математики Московского физико-технического института (141701, ГСП-4, Российская Федерация, г. Долгопрудный, Институтский переулок, 9), karlov.mipt@gmail.com

Дмитрий Феликсович Пастухов, кандидат физико-математических наук, доцент кафедры технологий программирования Полоцкого государственного университета (211440, Республика Беларусь, г. Новополоцк, ул. Блохина, 29), ORCID, dmitrij.pastuhov@mail.ru

Юрий Феликсович Пастухов, кандидат физико-математических наук, доцент кафедры технологий программирования Полоцкого государственного университета (211440, Республика Беларусь, г. Новополоцк, ул. Блохина, 29), ORCID, pulsar1900@mail.ru

Заявленный вклад авторов: **все авторы сделали эквивалентный вклад в подготовку публикации.**

Конфликт интересов: **авторы заявляют об отсутствии конфликта интересов.**

Все авторы прочитали и одобрили окончательный вариант рукописи.

Received / Поступила в редакцию 18.11.2024
Reviewed / Поступила после рецензирования 12.12.2024
Accepted / Принята к публикации 16.12.2024