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Solution of Boundary Value Problems for Certain Nonlinear Differential Equations Using the Bubnov-Galerkin Method

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Abstract
Introduction. This study investigates the possibility of numerically solving a boundary value problem with a nonlinear differential equation, continuous coefficients, and a right-hand side using the modified Bubnov-Galerkin method. In the problem formulation, the partial derivatives of the equation's coefficients are continuous functions of all arguments. The order of the nonlinear differential equation n is strictly less than the number of coordinate functions m.
Materials and Methods. To numerically solve the nonlinear boundary value problem, the modified Petrov-Galerkin method and the uniqueness property of decomposing a smooth function into a system of linearly independent polynomial basis functions on the interval $[-1,1]$ with a unit Chebyshev norm for each function in the system are used. The system of linear algebraic equations includes linearly independent boundary conditions. The matrix elements and the right-hand side of the system depend on the simple iteration index s. The coefficient vector of the solution decomposition into basis functions also depends on the index s. The inverse matrix of the system was computed using the Msimsl linear algebra library in Fortran.
Results. Sufficient conditions for the existence and uniqueness of the solution to the boundary value problem with a nonlinear differential equation using the simple iteration method have been formulated. When the sufficient conditions are met, the decomposition coefficients decrease absolutely as the basis function index increases.
Discussion and Conclusion. Three boundary value problems with a second-order nonlinear equation and one problem with a third-order equation were solved exactly. The analytical solutions were compared with numerical solutions, with the uniform norm of the difference having an order of 10^{-13}, 10^{-11}, 10^{-10}, 10^{-10}, respectively. The modified Bubnov-Galerkin method allows for solving each branch of a multivalued function in boundary value problems with nonlinear differential equations.

Keywords: hydrodynamics, mechanics, numerical methods, nonlinear differential equations, boundary value problems, Galerkin method

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Оригинальное теоретическое исследование

Решение краевых задач с нелинейными дифференциальными уравнениями методом Бубнова-Галеркина

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Аннотация

Введение. Исследуется возможность численного решения модифицированным методом Бубнова-Галеркина краевой задачи с нелинейным дифференциальным уравнением, непрерывными коэффициентами и правой частью. В постановке задачи частные производные коэффициентов уравнения являются непрерывными функциями по всем аргументам. Порядок нелинейного дифференциального уравнения n строго меньше числа координатных функций m.

Материалы и методы. Для численного решения нелинейной краевой задачи использован модифицированный метод Петрова-Галеркина и идея единственности разложения гладкой функции по системе линейно независимых базисных функций степенного вида на отрезке [–1,1] с единичной нормой Чебышева для каждой функции системы. В систему линейных алгебраических уравнений включены линейно независимые краевые условия. При этом элементы матрицы и правая часть системы зависят от индекса простой итерации s. От индекса s зависит и вектор коэффициентов разложения решения по базисным функциям. Обратная матрица системы находилась библиотекой линейной алгебры Msimsl на языке Fortran.

Результаты исследования. Сформулированы достаточные условия существования и единственности решения краевой задачи с нелинейным дифференциальным уравнением методом простой итерации. При выполнении достаточных условий коэффициенты разложения абсолютно уменьшаются с ростом номера базисной функции.

Обсуждение и заключение. Точно решены три краевых задачи с нелинейным уравнением второго порядка и одна задача с уравнением третьего порядка. Аналитические решения сравнены с численными решениями, равномерная норма разности имеет порядок 10^{-13}, 10^{-11}, 10^{-10}, 10^{-10} соответственно. Модифицированный метод Бубнова-Галеркина позволяет находить решение каждой ветви многозначной функции в краевых задачах с нелинейными дифференциальными уравнениями.

Ключевые слова: гидродинамика, механика, численные методы, нелинейные дифференциальные уравнения, краевые задачи, метод Галеркина

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Introduction. Various methods are known for solving boundary value problems on an interval with a nonlinear differential equation, such as Newton's method [1, p. 460]. Methods for solving boundary value problems in hydrodynamics and mechanics are also presented in [2, 3]. The methods for solving nonlinear boundary value problems share many similarities with those used for linear problems [1, p. 458]. In this study, the modified Bubnov-Galerkin method, previously proposed in [4, 5] for solving boundary value problems with linear differential equations, has been extended to the case of boundary value problems with nonlinear differential equations.

In this work, any branch (graph) of a smooth solution of a nonlinear differential equation is represented as a linear combination of polynomial-type basis functions (i. e., the numerical solution is expressed in functional form). All basis functions are defined on the interval [−1,1] with a unit Chebyshev norm.

The solution of the nonlinear boundary value problem is reduced to the simple iteration method. At each iteration, a system of linear algebraic equations (SLAE) is solved, where the matrix elements and the right-hand side coefficients depend on the iteration index s. For a third-order equation, the SLAE includes n–1 linearly independent boundary conditions and m–n+1 orthogonality conditions for the residual of the differential equation to the basis functions [1, 6] (n is the order of the ODE, m is the number of basis functions).

New results and ideas for solving boundary value problems, including high-accuracy solutions and problems in complex domains, have been obtained in [7–10].

Materials and Methods

Problem Statement. Let the unknown function $u(x)$, belonging to the class of functions that are n-times continuously differentiable on the interval $C^n[a, b]$, be the solution of a boundary value problem with a nonlinear differential equation of order n with variable coefficients $g_i(x,u(x),u'(x),...,u^{(i)}(x)) \in C[a,b], i=\overline{0,n}$

$$\begin{cases} L[u(x)] = f(x,u(x)),\ x\in(a,b), u(x)\in(c,d) \\ L[u(x)] \equiv \left(\sum_{i=0}^{n} g_i(x,u(x),u'(x),...,u^{(i)}(x))\frac{d^i}{dx^i}\right)u(x), \end{cases} \tag{1}$$

$$\sum_{i=0}^{n-1}\left(\alpha_\mu^i u^{(i)}(a) + \beta_\mu^i u^{(i)}(b)\right) = \gamma_\mu,\ \mu=\overline{1,n}. \tag{2}$$

In the boundary value problem (1)–(2), the given functions $g_i(x,u(x),u'(x),...,u^{(i)}(x)) \in C[a,b], i=\overline{0,n}$ are continuous on the interval $[a, b]$ with respect to all arguments. Each term of the nonlinear equation (1) can be expressed such that the coefficient $g_i(x,u(x),u'(x),...,u^{(i)}(x)), i=\overline{0,n}$, multiplying the derivative $\frac{d^i u(x)}{dx^i}$, depends only on derivatives of the function $u^{(p)}(x), p=\overline{0,i}$ up to order i at most.

For simplicity, the boundary conditions (2) at the points $x=a$, $x=b$ are given as linear forms with respect to the function and its derivatives up to order n–1 similar to a linear boundary value problem. To ensure the well-posedness of problem (1), the total number of boundary conditions must be equal to n. The coefficient matrices $\alpha_\mu^i, \beta_\mu^i,\ i=\overline{0,n-1}, \mu=\overline{1,n},$ as well as the values of $\gamma_\mu,\ \mu=\overline{1,n}$ are predefined. To uniquely determine a specific branch of the nonlinear solution of equation (1), one or more additional conditions can be imposed on the boundary conditions of type (2).

Assume that the coefficients $g_i(x,u(x),u'(x),...,u^{(i)}(x)) \in C[a,b], i=\overline{0,n}$ are continuously differentiable with respect to their variables $u^{(p)}$, i. e. $\frac{\partial g_i}{\partial u^{(p)}} \in C[c_i,d_i], p=\overline{0,i}$.

We generalize the Bubnov-Galerkin method, proposed in [4, 5] for solving boundary value problems with linear ordinary differential equations, to the case of a nonlinear differential equation. Suppose that any smooth function $u(x)$ or its specific branch (a solution of equation (1)) can be uniquely represented as a linear combination of polynomial basis functions [4, 5]:

$$\{\phi_i(x)\}_{i=0}^{m} = \left\{\left(\frac{2x-a-b}{b-a}\right)^i, x\in[a,b], i=\overline{0,m}\right\},\ \|\phi_i(x)\|_C = \max_{x\in[a,b]}|\phi_i(x)| = 1\,\forall i=\overline{0,m}, \tag{3}$$

$$u(x) = u(c) + \sum_{j=1}^{m}\phi_j(x)D_j = u(c) + \sum_{j=1}^{m}\left(\frac{2(x-c)}{b-a}\right)^j D_j, c=\frac{a+b}{2}. \tag{4}$$

The goal of the numerical algorithm is to determine the decomposition vector D_j for solving the boundary value problem (1)–(2) using the basis functions $\phi_j(x)$. We define the residual of the nonlinear equation (1) and take into account the representation of its solution using formula (4):

$$R(u(x)) = L[u(x)] - f(x,u(x)) = \left(\sum_{i=0}^{n} g_i(x,u(x),u'(x),...,u^{(i)}(x))\frac{d^i}{dx^i}\right)u(x) - f(x,u(x)) =$$

$$= \sum_{j=1}^{m} D_j \sum_{i=1}^{n} g_i(x,u(x),u'(x),...,u^{(i)}(x))\frac{d^i\phi_j(x)}{dx^i} + g_0(x,u(x))\cdot u(x) - f(x,u(x)) = \tag{5}$$

$$= \sum_{j=1}^{m} D_j \sum_{i=1}^{n} g_i(x,u(x),u'(x),...,u^{(i)}(x))\frac{d^i\phi_j(x)}{dx^i} - \overline{f(x,u(x))},\ \overline{f(x,u(x))} = f(x,u(x)) - g_0(x,u(x))\cdot u(x).$$

We require [1, 4–6] that the residual $R(u(x))$ be orthogonal to the maximum number of m–n+1 basis functions (where m is the number of basis coordinate functions and n is the order of the differential equation):

$$\langle R(u(x)), \phi_k(x)\rangle = 0 \Leftrightarrow k = \overline{0,m-n},$$

$$\left\langle \sum_{j=1}^{m} D_j \sum_{i=1}^{n} g_i(x,u(x),u'(x),...,u^{(i)}(x)) \frac{d^i \phi_j(x)}{dx^i} - \overline{f(x,u(x))}, \phi_k(x) \right\rangle = 0 \forall k = \overline{0,m-n} \Leftrightarrow$$
$$\begin{cases} \sum_{j=1}^{m} A_{k,j} D_j = F_k,\ F_k = \left\langle \overline{f(x,u(x))}, \phi_k(x) \right\rangle \\ A_{k,j} = \left\langle \sum_{i=1}^{n} g_i(x,u(x),u'(x),...,u^{(i)}(x)) \frac{d^i \phi_j(x)}{dx^i}, \phi_k(x) \right\rangle, k = \overline{0,m-n}. \end{cases} \tag{5}$$

Other linearly independent boundary conditions (2) for the system of linear algebraic equations (5) should be selected from the set of admissible functions of the original boundary problem (1) [11].

We will solve the system of linear algebraic equations (5), where the matrix coefficients and the coefficients of the right-hand side depend implicitly on the solution, using the simple iteration method. To do this, we will consider the implicit dependence of the solution on the decomposition vector $D_j^s, s = 0,1,2,..., j = \overline{1,m},$ (4) by the system of basis functions with iteration number *s*.

From (5) we obtain:

$$A_{k,j} = A_{k,j}\left(u^s\right) = A_{k,j}\left(u\left(D^s\right)\right), F_k = F_k\left(u^s\right) = F_k\left(u\left(D^s\right)\right),\ D = D^{s+1},$$
$$\sum_{j=1}^{m} A_{k,j}\left(D^s\right) \cdot D_j^{s+1} = F_k\left(D^s\right), j = \overline{1,m},\ k = \overline{0,m-n},\ D^{s+1} = A^{-1}\left(D^s\right) \cdot F\left(D^s\right), s = 0,1,2,... . \tag{6}$$

Let us denote the limiting values

$$\lim_{s\to\infty} A\left(D^s\right) = \overset{*}{A},\ \lim_{s\to\infty} F\left(D^s\right) = \overset{*}{F}, \lim_{s\to\infty} D^s = \overset{*}{D}.$$

We will write the limiting form of equation (6):

$$\overset{*}{A}\overset{*}{D} = \overset{*}{F}. \tag{7}$$

Theorem 1 (sufficient conditions for the existence and uniqueness of the solution $\overset{*}{D}$ of equation (7) in the simple iteration (6)).

Let the coefficients of equation (1) be continuous $g_i\,(x,u(x),u'(x),...,u^{(i)}(x)) \in C[a,b], i = \overline{0,n}$ and continuously differentiable with respect to all arguments, starting from the second $\frac{\partial g_i}{\partial u^{(p)}} \in C[c_p,d_p], p = \overline{0,i},\ i = \overline{1,n}$. Let the limiting matrix $\overset{*}{A}$ in (7), computed according to formulas (5), be non-singular. Let the conditions for a contraction mapping $q = \left\|\overset{*}{A}{}^{-1}\right\| \cdot \left\|\overset{*}{G}\right\| < 1$ be satisfied. Then, in the boundary value problem (1) with boundary conditions (2), there exists a unique solution. The residual norm decreases according to the formula $\left\|\delta D^s\right\| \le \left\|\delta D^0\right\| q^s / \left(1-q\right)$, where s is the iteration number in the algorithm (6).

Proof. Consider the increment of equation (6) in the vicinity of all its limiting values:

$$\overset{*}{A}\overset{*}{D} = \overset{*}{F} \Rightarrow \delta A \overset{*}{D} + \overset{*}{A} \delta D = \delta F,$$

where $\delta A^s = A^s - \overset{*}{A}$ is the increment of the matrix of dimension $m \times m$; $\overset{*}{A}$ is the matrix of dimension $m \times m$; $\delta D^{s+1} = D^{s+1} - \overset{*}{D}$ is the increment of the vector of dimension *m*; $\delta F^s = F^s - \overset{*}{F}$ is the increment of the vector of dimension *m*. Introduce an integer index $t = \overline{1,m}$ and consider the increment of the specified quantities δA^s, δF^s as a function of the increment of the component of the vector δD^s_t, using formula (5): $\delta D_t^s = D_t^s - \overset{*}{D}_t$:

$$\delta A_{k,j}(\delta D_t^s)\overset{*}{D} = \left\langle \sum_{j=1}^{m} \overset{*}{D}_j \sum_{i=1}^{n} \sum_{p=0}^{i} \frac{\partial g_i}{\partial u_p} \cdot \phi_t^{(p)}(x) \cdot \phi_j^{(i)}(x), \phi_k(x) \right\rangle \delta D_t^s, u_p \equiv u^{(p)}(x) \quad t = \overline{1,m},\ k = \overline{0,n-m}, j = \overline{1,m}.$$

In the last formula, summation over the index *p*, has been added because the coefficients of the nonlinear ODE $g_i\,(x,u(x),u'(x),...,u^{(i)}(x))$ depend on the function and its derivatives up to the *i*-th order, inclusively. The effective right-hand side of the nonlinear equation $\overline{f(x,u(x))} = f(x,u(x)) - g_0(x,u(x)) \cdot u(x)$ depends only on the function *u*(*x*):

$$\delta F_k(\delta D_t^s) = \left\langle \frac{\partial \overline{f(x,u(x))}}{\partial u} \phi_t(x), \phi_k(x) \right\rangle \delta D_t^s, t = \overline{1,m}.$$

We obtain $\delta D^{s+1} = \overset{*}{A}{}^{-1}\left(\delta F^s - \delta A^s \overset{*}{D}\right)$ and replace the last equation with a similar equation of simple iteration

$$\delta D^{s+1} = \left(A^s\right)^{-1}\left(\delta F^s - \delta A^s D^s\right) = \left(A^s\right)^{-1} G^s \delta D^s. \tag{8}$$

In equation (8)

$$A^s{}_{k,j} = \left\langle \sum_{i=1}^{n} g_i\left(x, u^s(x), u'^s(x), ..., u^{(i)s}(x)\right)\frac{d^i \phi_j(x)}{dx^i}, \phi_k(x) \right\rangle, k = \overline{0, m-n}, j = \overline{1, m}, \tag{9}$$

$$G^s{}_{k,t} = \left\langle \overline{\frac{\partial f(x,u(x))}{\partial u}}\phi_t(x), \phi_k(x)\right\rangle - \left\langle \sum_{j=1}^{m} D_j^s \sum_{i=1}^{n}\sum_{p=0}^{i} \frac{\partial g^s{}_i}{\partial u_p}\cdot \phi_t^{(p)}(x)\cdot \phi_j^{(i)}(x), \phi_k(x)\right\rangle,\ k = \overline{0, m-n}, t = \overline{1, m}, \tag{10}$$

$\delta D_t^{s+1} = \delta D_t^s = 0,\ t = \overline{m-n+1, m}$ the last rows of the matrix coefficients and the coefficients of the right-hand side in equation (5) are determined by the boundary conditions and are constant numbers. In formula (10), $\phi_t^{(p)}$ denotes the p-th order derivative of the basis coordinate function with index *t*. Similarly, for $\phi_j^{(i)}(x)$. Explicit formulas $\phi_j^{(i)}(x)$ are provided in [4, 5].

Let us estimate the right and left sides of equation (8)

$$\left\|\delta D^{s+1}\right\| \le \left\|\left(A^s\right)^{-1}\right\|\left\|G^s\right\|\left\|\delta D^s\right\| = q\left\|\delta D^s\right\|, q^s = \left\|\left(A^s\right)^{-1}\right\|\left\|G^s\right\| \le \left\|\left(\overset{*}{A}\right)^{-1}\right\|\left\|\overset{*}{G}\right\|. \tag{11}$$

by the norm in complete metric spaces [12]. If the compression parameter

$$q_s = \left\|\left(A^s\right)^{\overset{*}{-1}}\right\|\left\|G^s\right\| \le q = \left\|\left(\overset{*}{A}\right)^{-1}\right\|\left\|\overset{*}{G}\right\| < 1, s = 0, 1, 2, ...$$

for the mapping $\delta D^s \to \delta D^{s+1} = \left(A^s\right)^{-1} G^s \delta D^s$ is less than one, the solution to the boundary value problem (1)–(2) exists and is unique.

According to the work of A.N. Kolmogorov and S.V. Fomin [12, p. 87], the residual norm decreases according to the formula $\left\|\delta D^s\right\| \le \left\|\delta D^0\right\| q^s / (1-q)$, $\left\|\delta D^s\right\| = \left\|D^s - \overset{*}{D}\right\| = \rho\left(D^s, \overset{*}{D}\right)$.

Theorem 1 is proved.

Let us consider three examples of solving a nonlinear boundary value problem with a second-order equation and one example with a third-order equation.

Example 1.

$$\begin{cases} u\cdot u''(x) - u'\cdot u'(x) = u^2(x) = f(x,u), \\ u(0) = 1, u(1) = e^{1/2},\ x \in [0,1]. \end{cases} \tag{12}$$

The exact solution of the problem was obtained by considering the transformations:

$$u\cdot u''(x) - u'\cdot u'(x) = u^2(x) \Leftrightarrow \left(\frac{u'}{u}\right)' = \left(\ln(u(x))\right)'' = \frac{uu'' - \left(u'\right)^2}{u^2} = 1 \Leftrightarrow$$

$$\ln(u(x)) = \frac{x^2}{2} + C_1 x + C_2 \Leftrightarrow u(x) = \overline{C_2} e^{\frac{x^2}{2} + C_1 x},\ u(0) = 1 \Leftrightarrow \overline{C_2} = 1, u(1) = e^{1/2} \Leftrightarrow \frac{1}{2} + C_1 = \frac{1}{2} \Rightarrow C_1 = 0.$$

As a result, the exact solution of Example (1) is of the form $u(x) = e^{\frac{x^2}{2}}$.

Let us convert the condition of the example to a numerical algorithm (5)–(6). It is also necessary to specify the initial function — a solution that satisfies the Dirichlet boundary conditions, i. e., the given values of the function at the boundaries of the interval.

We obtain that

$$u^0(x) = \frac{u_0 + u_n}{2} + \phi_1(x) D_1^0 = \frac{u_0 + u_n}{2} + \phi_1(x)\left(\frac{u_n - u_0}{2}\right), D_2^0 = ... = D_n^0 = 0, \tag{13}$$

since

$$\phi_1(a) = -1, \phi_1(b) = 1, u^0(a) = \frac{u_0 + u_n}{2} - \left(\frac{u_n - u_0}{2}\right) = u_0, u^0(b) = \frac{u_0 + u_n}{2} + \left(\frac{u_n - u_0}{2}\right) = u_n$$

and the boundary conditions for the initial function $u^0(x)$ are satisfied. The coefficients and the right-hand side in Example 1 are as follows:

$$g_2(x,u,u',u'') = u, g_1(x,u,u') = -u', g_0(x,u) = 0, f(x,u) = u^2(x).$$

In the numerical algorithm, in addition to formulas (5)–(6), we use the detailed formulas from the work [5], which account for the Dirichlet boundary conditions for the second-order linear differential equation:

$$\sum_{j=1}^{m} a^s{}_{i,j} D_j^{s+1} = \overline{f_i}^s, \ i = \overline{0,m-1}. \tag{14}$$

Here, the elements of the matrix $a_{i,j}, i = \overline{0,m-1}, j = \overline{1,m}$ and the coefficients of the right-hand side $\overline{f_i}$ of the system of equations (14) are:

$$a^s{}_{i,j} = \begin{cases} \langle L^s \phi_j, \phi_i \rangle, j \equiv 1(\mathrm{mod}\,2), i = \overline{0,m-2}, \\ \langle L^s(\phi_j - 1), \phi_i \rangle, j \equiv 0(\mathrm{mod}\,2), i = \overline{0,m-2}, \\ 1, i = m-1, j \equiv 1(\mathrm{mod}\,2), \\ 0, i = m-1, j \equiv 0(\mathrm{mod}\,2), \end{cases}$$

$$\overline{f_i^s} = \begin{cases} \left\langle f(x,u^s) - L^s\left(\frac{u_a + u_b}{2}\right), \phi_i(x) \right\rangle, i = \overline{0,m-2}, \\ \frac{u_b - u_a}{2}, i = m-1. \end{cases}$$

However, the operator $L^s\phi_j$ for the matrix coefficients $a^s_{i,j}$ in (14) is nonlinear and is computed using formula (1), unlike in [5]. The absolute and relative Chebyshev vector norm of the residual for the problem:

$$\left\|u^{num} - u^{exact}\right\|_c = 1.072313895664201\mathrm{E}-013, \frac{\left\|u^{num} - u^{exact}\right\|_c}{\left\|u^{exact}\right\|_c} = 6.503912545562317\mathrm{E}-014.$$

The number of intervals for calculating the scalar product of two functions in (14) is n_1=20, the number of coordinate functions m=15, and the number of iterations n_1=30.

The scalar product of functions was computed using the formulas from [5]:

$$\langle y_1, y_2 \rangle = \int_a^b y_1(x) y_2(x) dx = 10h \sum_{i=0}^{n_1} y_1(x_i) y_2(x_i) C_i + O(h^{22}), n_1 = 20s, h = \frac{b-a}{n_1}, s \in N, \tag{15}$$

$$C_i = \begin{cases} \frac{1145302367137}{48426042384720}, \text{if } i = 0 \text{ or } i = n_1, \\ \frac{1145302367137}{24213021192360}, \text{ if } (i \equiv 0 \bmod 20) \text{ and } (0 < i < n_1), \\ \frac{335582304250}{1470076286679}, \text{ if } (i \equiv 1 \bmod 20) \text{ or } (i \equiv 19 \bmod 20), \\ \frac{-19467909708875}{41162136027012}, \text{ if } (i \equiv 2 \bmod 20) \text{ or } (i \equiv 18 \bmod 20), \\ \frac{8274871497250}{3430178002251}, \text{ if } (i \equiv 3 \bmod 20) \text{ or } (i \equiv 17 \bmod 20), \\ \frac{-413929922392625}{54882848036016}, \text{ if } (i \equiv 4 \bmod 20) \text{ or } (i \equiv 16 \bmod 20), \\ \frac{50652939811064}{2450127144465}, \text{ if } (i \equiv 5 \bmod 20) \text{ or } (i \equiv 15 \bmod 20), \\ -\frac{155790561130375}{3430178002251}, \text{ if } (i \equiv 6 \bmod 20) \text{ or } (i \equiv 14 \bmod 20), \\ \frac{286955364893000}{3430178002251}, \text{ if } (i \equiv 7 \bmod 20) \text{ or } (i \equiv 13 \bmod 20), \\ -\frac{502376261017625}{3920203431144}, \text{ if } (i \equiv 8 \bmod 20) \text{ or } (i \equiv 12 \bmod 20), \\ \frac{1704056522480500}{10290534006753}, \text{ if } (i \equiv 9 \bmod 20) \text{ or } (i \equiv 11 \bmod 20), \\ -\frac{1684005984173647}{9355030915230}, \text{ if } i \equiv 10 \bmod 20. \end{cases} \tag{16}$$

The results of the program (Example 1) are presented in Table 1.

Table 1

Numerical $u_i^{numerical}$ and exact u_i^{exact} solutions of the problem (12)

x_i	$u_i^{numerical}$	u_i^{exact}	$u_i^{numerical} - u_i^{exact}$
0.0000000000000E+000	1.00000000000000	1.00000000000000	0.00000E+000
5.0000000000000E–002	1.00125078157569	1.00125078157562	6.533608E–014
0.100000000000000	1.00501252085944	1.00501252085940	3.497831E–014
0.150000000000000	1.01131351922362	1.01131351922361	5.002183E–015
0.200000000000000	1.02020134002678	1.02020134002676	2.150048E–014
0.250000000000000	1.03174340749917	1.03174340749910	7.15878E–014
0.300000000000000	1.04602785990882	1.04602785990872	1.072313E–013
0.350000000000000	1.06316467213420	1.06316467213410	9.834472E–014
0.400000000000000	1.08328706767501	1.08328706767496	5.269027E–014
0.450000000000000	1.10655324549790	1.10655324549789	4.473201E–015
0.500000000000000	1.13314845306681	1.13314845306683	–1.17629E–014
0.550000000000000	1.16328744359303	1.16328744359302	1.449849E–014
0.600000000000000	1.19721736312188	1.19721736312181	6.498762E–014
0.650000000000000	1.23522112174449	1.23522112174439	1.029619E–013
0.700000000000000	1.27762131320499	1.27762131320489	1.004986E–013
0.750000000000000	1.32478475872893	1.32478475872887	6.074600E–014
0.800000000000000	1.37712776433597	1.37712776433596	1.600000E–014
0.850000000000000	1.43512219658388	1.43512219658387	5.959208E–015
0.900000000000000	1.49930250005679	1.49930250005677	2.719775E–014
0.950000000000000	1.57027380147662	1.57027380147660	1.296022E–014
1.00000000000000	1.64872127070013	1.64872127070013	4.737963E–017

The numerical solution of the problem (1)–(2) can be brought to the form (17) by transforming formula (4) with Dirichlet boundary conditions [4]:

$$u(x) = \left(\frac{u_a + u_b}{2}\right) + \sum_{j=1}^{m}\left[\left(\frac{(2x-a-b)}{b-a}\right)^j + \left(\frac{-1+(-1)^{j+1}}{2}\right)\right]\overset{*}{D}_j\,. \tag{17}$$

In (17), the limiting values $\overset{*}{D}_j$ of the decomposition vector of the solution by basis functions are used, and $\overset{*}{D}_j$ is the solution of the system of linear algebraic equations (14) on the last iteration.

Example 2.

$$\begin{cases} 2u \cdot u''(x) + 2u' \cdot u'(x) = x = f(x,u), \\ u(1) = 1, u(2) = 2,\ x \in [1,2]. \end{cases} \tag{18}$$

Since

$$\left(u^2(x)\right)' = 2uu',\ \left(u^2(x)\right)'' = 2\left(u'\right)^2 + 2uu'' = x \Rightarrow u^2(x) = \frac{x^3}{6} + C_1 x + C_2 \Leftrightarrow$$

$$u(x) = \pm\sqrt{\frac{x^3}{6} + C_1 x + C_2}, \begin{cases} u(1) = 1 \Leftrightarrow 1 = \dfrac{1}{6} + C_1 + C_2 \\ u(2) = 2 \Leftrightarrow 4 = \dfrac{4}{3} + 2C_1 + C_2 \end{cases} \Leftrightarrow \begin{cases} C_1 + C_2 = \dfrac{5}{6} \\ 2C_1 + C_2 = \dfrac{8}{3} \end{cases} \Leftrightarrow C_1 = \frac{11}{6}, C_2 = -1.$$

it follows that $u(x) = \sqrt{\dfrac{x^3 + 11x - 6}{6}}$ is the exact solution of the boundary value problem (18).

Note that in this case, the boundary conditions (18) unambiguously choose one branch of the solution; otherwise, additional conditions can be set to select one branch.

The number of intervals for calculating the scalar product of two functions in (18) is n_1=100, the number of coordinate functions m=18, and the number of iterations n_1=30, $g_2(x,u,u',u'') = 2u, g_1(x,u,u') = 2u', g_0(x,u) = 0, f(x,u) = x.$

Table 2

Numerical $u_i^{numerical}$ and exact u_i^{exact} solutions of the problem (18)

x_i	$u_i^{numerical}$	u_i^{exact}	$u_i^{numerical} - u_i^{exact}$
1.00000000000000	1.00000000000000	1.00000000000000	0.0000000000E+000
1.05000000000000	1.05732563573	1.05732563574331	–9.187983707E–012
1.10000000000000	1.11287914887	1.1128791488746	3.6390890301E–012
1.15000000000000	1.16696722317	1.16696722319009	–1.377342684E–011
1.20000000000000	1.21983605454	1.21983605455815	–1.174038644E–011
1.25000000000000	1.27168687184	1.27168687183599	4.801492536E–012
1.30000000000000	1.32268665978	1.32268665979513	–8.256284544E–012
1.35000000000000	1.37297578272	1.37297578274345	–2,059930004E–011
1.40000000000000	1.42267353949	1.42267353950230	–5.676792369E–012
1.45000000000000	1.47188229828	1.47188229828339	5.4383164638E–012
1.50000000000000	1.52069063256	1.52069063257455	–8.254952277E–012
1.55000000000000	1.56917573902	1.56917573904264	–1.900479773E–011
1.60000000000000	1.61740532952	1.61740532953246	–6.780798145E–012
1.65000000000000	1.66543913128	1.66543913128040	3.2989166953E–012
1.70000000000000	1.71333009078	1.71333009078811	–7.917888567E–012
1.75000000000000	1.76112535043	1.76112535045067	–1.482725053E–011
1.80000000000000	1.80886704873	1.80886704873520	–2.779998453E–012
1.85000000000000	1.85659298177	1.85659298178141	–1.615374500E–012
1.90000000000000	1.90433715501	1.90433715502271	–6.362910198E–012
1.95000000000000	1.95213024668	1.95213024667925	6.6588956570E–012
2.00000000000000	2.00000000000000	2.00000000000000	0.0000000000E+000

The Chebyshev vector norm (absolute and relative norms) of the residual for the problem in Example (18) are:

$$\left\|u^{num} - u^{exact}\right\|_c = 2.059930004350008\mathrm{E}-011, \frac{\left\|u^{num} - u^{exact}\right\|_c}{\left\|u^{exact}\right\|_c} = 1.029965002175004\mathrm{E}-011.$$

Note. If the boundary conditions in Example (18) are chosen as $u(1) = -1$, $u(2) = -2$, and the exact solution of the problem is $u(x) = -\sqrt{\dfrac{x^3+11x-6}{6}}$, the program gives the same Chebyshev vector norm of the residual due to the symmetry of Example (18) with respect to the transformation $(x,u) \rightarrow (x,-u)$.

Therefore, the algorithm using formulas (1), (5), (6), (13), (15), (16), (17) for the nonlinear Dirichlet boundary value problem with a second-order equation finds the solution for each branch of the multi-valued solution function. If the boundary conditions for the branches coincide, an additional condition can be selected, for example, the value of the function at the midpoint of the interval $u(c) = u\left(\dfrac{a+b}{2}\right)$ and this condition can be included in the system of linear algebraic equations (14), thereby reducing the number of orthogonality conditions for the residual of the ODE with respect to the basis functions by 1.

Example 3.

$$\begin{cases} \cos(u)\cdot u''(x) - \sin(u)u'\cdot u'(x) = 2 = f(x,u), \\ u(0) = 0, u(1) = \pi/6,\ x \in [0,1]. \end{cases} \tag{19}$$

Since

$$\left(\sin(u)\right)^{'}=\cos(u)u^{'},\ \left(\sin(u)\right)^{''}=\cos(u)u^{''}-\sin(u)\left(u^{'}\right)^{2}=2\Leftrightarrow\sin(u(x))=x^{2}+C_{1}x+C_{2}\Rightarrow$$

$$u(x)=\arcsin\left(x^{2}+C_{1}x+C_{2}\right),\begin{cases}u(0)=0\Leftrightarrow 0=\arcsin\left(C_{2}\right)\\ u(1)=\pi/6\Leftrightarrow\dfrac{1}{2}=1+C_{1}+C_{2}\end{cases}\Leftrightarrow\begin{cases}C_{2}=0\\ C_{1}+C_{2}=-\dfrac{1}{2}\end{cases}\Leftrightarrow C_{1}=-\frac{1}{2},C_{2}=0.$$

The function $u(x)=\arcsin\left(x^{2}-\dfrac{x}{2}\right)$ is the exact solution to the problem (19).

In Example (19) $g_{2}(x,u,u^{'},u^{''})=\cos(u), g_{1}(x,u,u^{'})=-\sin(u)u^{'}, g_{0}(x,u)=0, f(x,u)=2.$ The number of intervals for the scalar product of two functions is n_1=120, the number of coordinate functions is m=17, and the number of iterations is n_1=30.

Table 3

Numerical $u_i^{numerical}$ and exact u_i^{exact} solutions of the problem (19)

x_i	$u_i^{\text{numerical}}$	u_i^{exact}	$u_i^{\text{numerical}}-u_i^{\text{exact}}$
0.0000000E+000	0.0000000000E+000	0.00000000000E+000	0.00000E+000
4.16666666E–002	–1.90983832E–002	–1.9098383217E–002	–3.18418E–013
8.33333333E–002	–3.47292032E–002	–3.47292030514E–002	–2.030325E–010
0.125000000000	–4.68921831E–002	–4.68921831332E–002	–1.491510E–011
0.166666666666	–5.55841730E–002	–5.5584173280E–002	2.1348351E–010
0.208333333333	–6.08013439E–002	–6.08013437340E–002	–2.052915E–010
0.250000000000	–6.25407621E-002	–6.25407617964E–002	–3.182648E–010
0.291666666666	–6.08013435E–002	–6.0801343734E–002	1.7451315E–010
0.333333333333	–5.55841729E–002	–5.5584173280E–002	3.3926163E–010
0.375000000000	–4.68921832E–002	–4.6892183133E–002	–1.666960E–010
0.416666666666	–3.47292035E–002	–3.47292030514E–002	–5.013826E–010
0.458333333333	–1.90983833E–002	–1.90983832179E–002	–8.629465E–011
0.500000000000	4.30824846E–010	0.00000000000E+000	4.3082484E–010
0.541666666666	2.25713611E–002	2.25713609537E–002	2,0268367E–010
0.583333333333	4.86302760E–002	4.86302764989E–002	–4.379981E–010
0.625000000000	7.82046914E–002	7.82046919347E–002	–4.480854E–010
0.666666666666	0.1113410145	0.111341014340964	2.1420220E–010
0.708333333333	0.1481103594	0.148110359030227	3.9991367E–010
0.750000000000	0.1886163853	0.188616386175404	–2.500614E–010
0.791666666666	0.2330054316	0.233005432127055	–4.563276E–010
0.833333333333	0.2814800734	0.281480073230845	2.1672891E–010
0.875000000000	0.3343179941	0.334317994036368	1.4096033E–010
0.916666666666	0.3918993389	0.391899339315036	–3.222679E–010
0.958333333333	0.4547481928	0.454748192610442	2.4028019E–010
1.000000000000	0.523598775598299	0.523598775598299	2.8514517E–017

The Chebyshev vector norm of the residual for the problem in Example (19) is

$$\left\|u^{num}-u^{exact}\right\|_{c}=5.602360454820782\mathrm{E}-010\ .$$

Example 4 (with a nonlinear third-order differential equation).

$$\begin{cases}-\sin(u)\cdot u^{'''}(x)-3\cos(u)\cdot u^{'}\cdot u^{''}(x)+\sin(u)\cdot\left(u^{'}\right)^{2}\cdot u^{'}=6=f(x,u),\\ u(0)=\pi/3,u(1)=\pi/2,u^{'}(1)=0\,,x\in[0,1]\rightarrow u\in[\pi/3,\pi/2].\end{cases}\quad(20)$$

Since

$$\left(\cos(u)\right)^{'}=-\sin(u)u^{'},\ \left(\cos(u)\right)^{''}=-\sin(u)u^{''}-\cos(u)\left(u^{'}\right)^{2},\left(\cos(u)\right)^{'''}=-\sin(u)u^{'''}-3\cos(u)u^{'}u^{''}+\sin(u)\left(u^{'}\right)^{3},$$

$$\cos(u)=x^{3}+C_{1}x^{2}+C_{2}x+C_{3},u(x)=\arccos\left(x^{3}+C_{1}x^{2}+C_{2}x+C_{3}\right),$$

$$\left(\cos(u)\right)^{'}=-\sin(u)u^{'}=3x^{2}+2C_{1}x+C_{2},\ u(0)=\pi/3\Leftrightarrow\cos(\pi/3)=1/2=C_{3},$$

$$\begin{cases}u(1)=\pi/2\Leftrightarrow\cos(\pi/2)=0=1+C_{1}+C_{2}+1/2\\ u^{'}(1)=0=\dfrac{\left(3x^{2}+2C_{1}x+C_{2}\right)_{x=1}}{-\sin(u=\pi/2)}\Leftrightarrow 3+2C_{1}+C_{2}=0\end{cases}\Leftrightarrow\begin{cases}C_{1}+C_{2}=-3/2\\ 2C_{1}+C_{2}=-3\end{cases}\Leftrightarrow C_{1}=-\frac{3}{2},C_{2}=0.$$

The function $u(x)=\arccos\left(x^{3}-\frac{3x^{2}}{2}+\frac{1}{2}\right)=\arccos\left(\frac{2x^{3}-3x^{2}+1}{2}\right)$ is the exact solution to the problem (20).

The coefficients of the differential equation in Example (20) are $g_{3}(x,u,u^{'},u^{''}u^{'''})=-\sin(u), g_{2}(x,u,u^{'},u^{''})=-3\cos(u)u^{'}, g_{1}(x,u,u^{'})=\sin(u)\left(u^{'}\right)^{2}, g_{0}(x,u)=0, f(x,u)=6.$

For the third-order equation, the formulas of the algorithm from the work [5] need to be modified, taking into account the iteration index:

$$\sum_{j=1}^{m}a_{i,j}^{s}D_{j}^{s+1}=\overline{f_{i}^{s}},\ i=\overline{0,m-1}. \tag{21}$$

where the elements of the matrix $a_{i,j}^{s}, i=\overline{0,m-1}, j=\overline{1,m},\ s=0,1,2,...$ and the coefficients of the right-hand side $\overline{f_{i}^{s}}$ of the system of equations (21), considering the nonlinear form of the differential operator L^s, computed using formula (1), are:

$$L^{s}[u^{s}(x)]\equiv\left(\sum_{i=0}^{n}g_{i}(x,u^{s}(x),\left(u^{'}(x)\right)^{s},...,\left(u^{(i)}(x)\right)^{s})\frac{d^{i}}{dx^{i}}\right)u^{s}(x),$$

$$L^{s}[\phi_{j}]\equiv\left(\sum_{i=0}^{n}g_{i}(x,u^{s}(x),\left(u^{'}(x)\right)^{s},...,\left(u^{(i)}(x)\right)^{s})\frac{d^{i}}{dx^{i}}\right)\phi_{j},$$

$$a_{i,j}^{s}=\begin{cases}\left\langle L^{s}\phi_{j},\phi_{i}\right\rangle,\ \text{if}\ j\equiv1\left(\operatorname{mod}2\right),i=\overline{0,m-3},\\ \left\langle L^{s}\left(\phi_{j}-1\right),\phi_{i}\right\rangle,\ \text{if}\ j\equiv0\left(\operatorname{mod}2\right),i=\overline{0,m-3},\\ 1,\ \text{if}\ i=m-2,\ j\equiv1\left(\operatorname{mod}2\right),\\ 0,\ \text{if}\ i=m-2,\ j\equiv0\left(\operatorname{mod}2\right),\\ j,\ \text{if}\ i=m-1,\end{cases}$$

$$\overline{f_{i}^{s}}=\begin{cases}\left\langle f(x)-L^{s}\left(\frac{u_{0}+u_{n}}{2}\right),\phi_{i}(x)\right\rangle,\ \text{if}\ i=\overline{0,m-3},L^{s}\left(\frac{u_{0}+u_{n}}{2}\right)=\left(\frac{u_{0}+u_{n}}{2}\right)g_{0}(x),\\ \dfrac{u_{n}-u_{0}}{2},\ \text{if}\ i=m-2,\\ 0,\ \text{if}\ i=m-1.\end{cases}$$

The last row of coefficients for a_{ij}^{s} в (21) in (21) is obtained by differentiating formula (4), since at the right end of the interval in Example (20) $u^{'}(b=1)=0$:

$$\left(u^{'}(x)\right)^{s}=\sum_{j=1}^{m}\phi_{j}^{'}(x)D_{j}^{s}=\sum_{j=1}^{m}\frac{2j}{(b-a)}\left(\frac{(2x-a-b)}{b-a}\right)_{x=b}^{j-1}D_{j}^{s}=0\Leftrightarrow D_{1}^{s}+2D_{2}^{s}+3D_{3}^{s}+...+mD_{m}^{s}=0.$$

The number of intervals for the scalar product of two functions in (20) is n_1=100, the number of coordinate functions m=17, and the number of iterations n_1=30.

Table 4

Numerical $u_i^{numerical}$ and exact u_i^{exact} solutions of the problem (20)

x_i	$u_i^{numerical}$	u_i^{exact}	$u_i^{numerical}-u_i^{exact}$
0.00000000E+000	1.047197551196	1.047197551196	–3.118165448E–016
5.00000000E–002	1.05137830714	1.051378307162	–1.8182033E–011

End of table 4

x_i	$u_i^{numerical}$	u_i^{exact}	$u_i^{numerical} - u_i^{exact}$
0.1000000000000	1.06328930398	1.063289303975	8.411912659E–012
0.1500000000000	1.08193034985	1.081930349812	4.527497062E–011
0.2000000000000	1.10626929727	1.106269297375	–1.03325620E–010
0.2500000000000	1.13528395573	1.135283955730	3.742097994E–012
0.3000000000000	1.16799174328	1.167991743151	1.319257321E–010
0.3500000000000	1.20346614669	1.203466146733	–3.93724372E–011
0.4000000000000	1.24084180881	1.240841808965	–1.47600610E–010
0.4500000000000	1.27931122393	1.279311223897	3.508758333E–011
0.5000000000000	1.31811607179	1.318116071652	1.431814589E–010
0.5500000000000	1.35653572032	1.356535720348	–2.54165034E–011
0.6000000000000	1.39387479319	1.393874793328	–1.33199911E–010
0.6500000000000	1.42945115362	1.429451153606	1.359530738E–011
0.7000000000000	1.46258526518	1.462585265086	1.018387917E–010
0.7500000000000	1.49259163483	1.492591634860	–2.76735182E–011
0.8000000000000	1.51877286350	1.518772863566	–6.40170944E–011
0.8500000000000	1.54041665402	1.540416653986	3.963527726E–011
0.9000000000000	1.55679586942	1.556795869421	–1.078461321E–013
0.9500000000000	1.56717131885	1.567171318855	–3.809447874E–012
1.000000000000	1.570796326794	1.570796326794	–6.125742274E–017

The Chebyshev vector norm (absolute and relative norms) of the residual for the problem in Example (20) are

$$\left\| u^{num} - u^{exact} \right\|_c = 1.522944525480727\text{E}-010\,, \frac{\left\| u^{num} - u^{exact} \right\|_c}{\left\| u^{exact} \right\|_c} = 1.454304895709992\text{E}-010.$$

Note 2. It should be noted that in the systems of linear algebraic equations (14) and (21) for the 4 examples, the library of linear algebra msimsl in FORTRAN was used for the computation of D_j^{s+1}.

Note 3. A comparison of the results of numerical solutions of the examples in this paper with the solutions of examples [4], [5] shows that the accuracy of solving boundary value problems is achieved greater than 10^{-13}–10^{-14} if a small number of intervals were used in the scalar product formula (n_1=$\overline{20.50}$ in the first example of this paper, n_1=20) and the rounding error is not it managed to grow due to fewer calculations. If the number of intervals is large (n_1=100) in the second, third and fourth examples of this work, then the accuracy of calculations is low (10^{-10}) due to the increase in rounding error and its effect on the overall error.

Discussion and Conclusion. A numerical solution algorithm for the boundary value problem on the interval with a nonlinear differential equation of order n, modified by the Bubnov-Galerkin method, is proposed. The possibility of decomposing the smooth solution of the nonlinear problem into a system of linearly independent basis functions with a unit Chebyshev norm on the interval [–1,1] is assumed. The number of basis functions m is greater than the order of the differential equation n. Formulas for the elements of the matrix and the coefficients of the right-hand side in the system of linear algebraic equations of the second and third orders are obtained. The systems of linear algebraic equations (14) or (21) are solved sequentially using the simple iteration method, with the number of iterations n_1=30. The theorem — sufficient conditions for the existence and uniqueness of the solution of the boundary value problem with the nonlinear ODE using the simple iteration method is proven.

Four nonlinear boundary value problems are solved analytically. The Chebyshev norm of the difference between the exact and numerical solutions in the solved examples has an order of magnitude, 10^{-13}, 10^{-11}, 10^{-10}, 10^{-10} respectively. This accuracy of the solution is intermediate between single 10^{-8} and double precision 10^{-15}, and is also comparable to the accuracy 10^{-11} of the solution of the linear boundary value problem in the work [4].

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