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Estimation of the Unidirectional Traffic Flow Velocity Limit with High Computational Efficiency

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Abstract

Introduction. In the modern development of intelligent transportation systems (ITS), an urgent task is the accurate estimation of the velocity limit of traffic flow on a highway. Despite existing solutions to this problem based on statistical mechanics methods and stochastic models, gaps remain in adapting these theories to real road segments of limited length. The traditional thermodynamic limit formula, used to calculate the average velocity of traffic flow, becomes inaccurate for small road segment lengths, limiting its applicability in practical traffic monitoring tasks. The aim of this study is a comparative analysis of various approaches to estimating the average velocity limit of traffic flow.

Materials and Methods. The study was conducted using the method of statistical mechanics and a stochastic model on a one-dimensional finite lattice. Numerical experiments with various parameter values (number of cells, traffic density, and movement probability) were used for analysis.

Results. The study revealed significant discrepancies between the results obtained using the statistical mechanics method and other approaches when the road segment length was small. The efficiency of the second and third approaches was confirmed for limited road segments, where they demonstrated greater accuracy and applicability.

Discussion and Conclusion. The research results have practical significance for the development of intelligent traffic management systems, especially for short road segments. The proposed approaches can be successfully integrated into modern monitoring systems to improve their accuracy. The theoretical significance of this work lies in advancing the methodology for traffic flow estimation while accounting for the specific conditions of real-world environments.

Keywords: traffic flows, thermodynamic limit, exclusion processes, asymptotic behavior of average velocity, stationary solutions, probabilistic traffic model, queuing systems

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Оригинальное теоретическое исследование

Оценка предельной скорости однонаправленного транспортного потока с высокой вычислительной эффективностью

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Аннотация

Введение. В современных условиях развития интеллектуальных транспортных систем (ITS) возникает актуальная задача точной оценки предельной скорости транспортного потока на магистрали. Несмотря на существующие решения данной проблемы, основанные на методах статистической механики и стохастических моделях, остаются

пробелы в адаптации этих теорий для реальных сегментов дорог с ограниченной протяженностью. Традиционная формула термодинамического предела, используемая для расчета средней скорости транспортного потока, становится некорректной при малых значениях длины дорожной полосы, что ограничивает ее применимость в практических задачах мониторинга транспорта. Целью настоящего исследования является сравнительный анализ различных подходов к оценке средней предельной скорости транспортного потока.

Материалы и методы. Исследование проведено на основе метода статистической механики и стохастической модели на одномерной конечной решетке. Для анализа использовались численные эксперименты с различными значениями параметров (число клеток, плотность потока, вероятность движения).

Результаты исследования. Проведенное исследование показало значительные расхождения между результатами метода статистической механики и другими подходами при малых значениях длины дорожной полосы. Эффективность второго и третьего подходов была подтверждена для ограниченных сегментов дорог, где они демонстрируют большую точность и применимость.

Обсуждение и заключение. Результаты исследования имеют практическое значение для разработки интеллектуальных систем управления транспортными потоками, особенно на коротких участках дорог. Предложенные подходы могут быть успешно интегрированы в современные системы мониторинга для повышения их точности. Теоретическая значимость работы заключается в развитии методологии оценки транспортных потоков с учетом специфики реальных условий.

Ключевые слова: транспортные потоки, термодинамический предел, случайные процессы с запретами, асимптотическое поведение средней скорости, стационарные решения, вероятностная модель трафика, системы массового обслуживания

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Introduction. The complexity of traffic flow modeling arises from the need to account for numerous factors, such as driver behavior, weather conditions, and the technical characteristics of vehicles and infrastructure. Despite significant progress in the development of mathematical traffic flow models, existing approaches often face a number of limitations that reduce their applicability in real-world conditions.

Mathematical models based on physical concepts traditionally describe physical media using methods from solid mechanics, hydrodynamics, and gas dynamics. These approaches are formulated as systems of differential equations, which impose smoothness requirements on their solutions. For example, the Navier-Stokes equations in hydrodynamics assume that the velocity and pressure of a fluid change smoothly, allowing for the description of fluid flow across a wide range of conditions. However, such models, based on differential equations, often lack the flexibility needed to describe complex and nonlinear processes, such as traffic flows, where abrupt changes in density and velocity are observed.

One of the fundamental properties of many controlled, interacting many-body systems is the emergence of shocks. A shock in a system of classical flowing particles represents a sudden transition from a low-density region to a high-density region. A well-known example of a shock is the onset of a traffic jam on a highway (Fig. 1), where incoming vehicles (almost freely flowing particles in a low-density regime) must rapidly decelerate over a short distance, subsequently becoming part of a high-density congested area. A remarkable feature of such shocks is their stability over long periods, meaning they remain localized over distances comparable to the size of individual particles. In a certain sense, these shocks can be viewed as soliton-like collective excitations of the particle system.

The difference between fluid (or gas) flow and traffic flow is too significant to employ a continuous approach and accurately describe traffic flow using macroscopic models [1]. Inaccuracies arise in situations where, for example, very few vehicles are present in the region affecting the movement of a given vehicle, compared to the vast number of particles in thermodynamic problems. Unlike mathematical physics, where molecules obey physical laws that are typically simple and constant, drivers exhibit diverse behavior and adapt their actions, introducing a human factor. While molecules move chaotically, drivers share similar goals (same direction, similar desired speeds) and generally prefer to avoid frequent acceleration and deceleration, making their behavior more varied than that of particles in a fluid or gas.

To address these challenges, this study proposes an evaluation of the velocity limit of unidirectional traffic flow based on various modelling approaches.

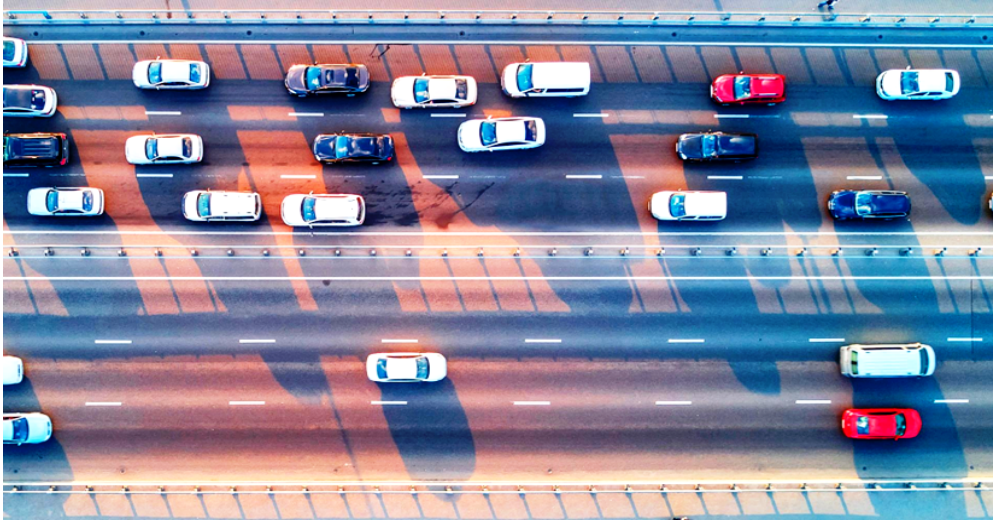


Fig. 1. Illustration of the Discontinuity (Stepwise Change) in Traffic Flow Density

Lighthill-Whitham-Richards (LWR) Model. One of the most popular approaches to traffic modelling since the mid-1950s has been the network hydrodynamic model and its various modifications, which liken vehicle flow to the motion of a continuous fluid. Hydrodynamic-based models (Payne [2], Kerner-Konhauser [3], Aw-Rascle [4], Zhang [5]) account for additional factors such as driver behavior, flow heterogeneity, phase transitions, and others.

In the 1955 work by Lighthill and Whitham [6] and the 1956 work by Richards [7], the dependence of flow functions on density was likened to fluid flow, where the density $\rho(x,t)$, is defined as the number of vehicles per unit road length, and the flow rate $q(x,t)$, represents the number of vehicles crossing a given section x , where $x \in G$. Considering the law of mass conservation, the number of vehicles remains constant within a closed system. In the case of an open system, we have $q_x + \rho_t = 0$ and $q_x + \rho_t = g(x,t)$. The function $g(x,t)$ represents the rate of vehicles entering or exiting the system. The velocity of traffic flow at a point x at time t is denoted as $v(x,t)$, and in the LWR model, it is assumed that velocity depends only on density. This model corresponded well with real-world observations, particularly at low traffic flow intensities.

Thus:

$$\begin{cases} \frac{\partial \rho}{\partial t} + \frac{\partial(\rho v(\rho))}{\partial x} = 0, \\ v(t, \rho) = F(\rho) \end{cases} \quad (1)$$

where $F(\rho)$ is a non-increasing function, not necessarily convex.

Mathematically, this model is analogous to the equation of motion for a compressible fluid, where the fluid density corresponds to traffic density, and the particle flow $\rho v(\rho)$ is analogous to mass transport flow in hydrodynamics. Although hydrodynamic-type models are widely used to describe traffic flows, they have several limitations that become apparent when analyzing real-world data:

1. *Lack of Discreteness in Vehicle Representation.* Hydrodynamic-type models assume that traffic flow behaves as a continuous medium, whereas in reality, vehicles are discrete entities. This limits the models' ability to describe microscopic phenomena such as gaps or density jumps, which are characteristic of real road conditions.

2. *Inaccurate Representation of Abrupt Density Changes.* The equations of hydrodynamic-type models are based on the assumption of density function smoothness, making it impossible to model discrete or sudden changes, such as unexpected traffic jams.

3. *Limitations in Modelling Complex Vehicle Interactions.* These models assume that a vehicle's speed depends solely on local density, ignoring factors such as individual driver characteristics, perception delays, and long-range interactions.

4. *Limited Adaptability to Real-World Conditions.* The models do not account for external influences such as weather conditions, accidents, or variable speed limits.

5. *Inability to Describe Phase Transitions.* Real traffic flows exhibit phase transitions between free-flowing traffic, stable flow, and congestion. These phenomena are difficult to accurately describe within hydrodynamic-type models without additional assumptions or modifications.

Gas Dynamic Models. Researchers have attempted to address the shortcomings of hydrodynamic models by introducing additional functions into the right-hand side of the equations. One of the most successful models in this regard is the quasi-gas-dynamic (QGD) model, developed under the guidance of Academician B.N. Chetverushkin [8].

The QGD model for traffic flows and its numerical implementation are discussed in detail in [9]. It is based on analogies between vehicle behavior and gas movement, allowing the application of gas dynamics approaches and methods. The

main advantage of this model is that, unlike gases where molecules are distributed in three-dimensional space, in traffic flows, the particles (vehicles) have finite sizes and move along constrained road lanes. In the one-dimensional case, the QGD system of equations for describing vehicular traffic flows takes the following form:

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho V}{\partial x} = \frac{\partial}{\partial x} \left(\frac{\tau}{2} \frac{\partial (\rho V^2 + P)}{\partial x} \right) + F_\rho, \quad (2)$$

$$\frac{\partial \rho V}{\partial t} + \frac{\partial \rho V^2}{\partial x} = f - \text{grad} P + \frac{\partial}{\partial x} \left(\frac{\tau}{2} \frac{\partial (\rho V^3 + P V)}{\partial x} \right) + F_V. \quad (3)$$

In this second-order model $\rho \left[\frac{\text{auto}}{\text{km} \times \text{lane}} \right]$ represents the flow density and $V \left[\frac{\text{km}}{\text{h}} \right]$ represents the vehicle speed.

Equation (2) defines the flow density, while equation (3) defines its speed.

For the case of low density, when the distances between vehicles are significant and their interaction is minimal, a simplified model can be used, analogous to incompressible fluid flow. In this case, the density of the traffic flow is considered constant, and changes only occur in the speed and direction of the flow. However, this approach is rarely used in traffic models, as density plays a key role in the formation of traffic jams and other phenomena.

Connection between Microscopic and Macroscopic Models. In more complex microscopic models, where a particle corresponds to a vehicle, the ratio of the number of particles to the number of cells in a section of the grid is a variable quantity. The average speed of the particles and the flow intensity also depend on the location of the grid section. The local state of a section can be characterized by the particle flow density and flow intensity. When the distribution of one of these characteristics is given on the grid, the distribution of the other characteristics is typically studied through simulation modelling.

In a macroscopic model of vehicle traffic flow, the relationship between density, speed, and intensity is described using partial differential equations (equations of mathematical physics). The relationship between the distribution of these characteristics and their dynamics is determined by these differential equations, which are usually solved using numerical methods under given boundary conditions.

Both microscopic and macroscopic traffic models are similar to or directly correspond with statistical physics models. In microscopic models of statistical physics, particles correspond to molecules or elementary particles, while in macroscopic models, distributions of characteristics such as pressure, temperature, flow velocity, and gas or liquid density are defined.

Microscopic models of traffic flows, described by random processes of various types [10], have advantages over macroscopic models as they allow the individual behavior of each vehicle to be considered. This is especially important for analyzing complex situations such as traffic jams, flow merging, or intersections. These models provide a more detailed description of the movement dynamics at the level of individual objects, making them useful in developing real-time traffic management strategies and creating intelligent transportation systems.

Probabilistic Traffic Model on a One-Dimensional Lattice. This work examines the implementation of a microscopic model on a single traffic lane. To obtain stationary solutions, we will investigate it on a closed ring lattice.

Let there be a ring lattice containing N cells, with $M < N$ particles (Fig. 2). Time is discrete. At each moment $t = 0, 1, 2, \dots$ each particle occupies one of the cells. Each cell can contain no more than one particle at a time. The cells are numbered $0, 1, \dots, N-1$ with the numbering directed along the direction of particle movement modulo N .

If at time $t = 0, 1, 2, \dots$ a particle is in cell i and cell $i + 1 \pmod{N}$ is free, then at time $t + 1$ the particle will, with probability $0 < p < 1$ be in cell $i + 1$ and with probability $1 - p$ the particle will remain in cell. If cell $i + 1$ is occupied, the particle cannot move.

We introduce the concept of the state space of a ring lattice $S = \{s_0, s_1, \dots, s_{C_N^M - 1}\}$, where the total number of possible states C_N^M is the number of combinations of N elements out of M . Each state s_j , $j = 0, 1, \dots, C_N^M - 1$ can be represented by a vector of length $N(x_0, x_1, x_2, \dots, x_{i-1}, x_i, x_{i+1}, \dots, x_{N-1})$, where element $x_i = 1$ if cell i is occupied and $x_i = 0$, if cell i is free.

Thus $\sum_{i=0}^{N-1} x_i = M$.

The transition matrix P has dimensions $C_N^M \times C_N^M$.

Let's consider an example for $N = 4$, $M = 2$.

States of the system:

$$S = \{(1, 1, 0, 0), (1, 0, 1, 0), (1, 0, 0, 1), (0, 1, 1, 0), (0, 1, 0, 1), (0, 0, 1, 1)\}.$$

Let's consider the initial state $s_0 = (1, 1, 0, 0)$. The possible transitions are:

- $(1, 1, 0, 0) \rightarrow (1, 1, 0, 0)$ with probability $1 - p$;
- $(1, 1, 0, 0) \rightarrow (1, 0, 1, 0)$ with probability p .

Now consider the initial state $s_1 = (1,0,1,0)$. The possible transitions are:

- $(1,0,1,0) \rightarrow (1,0,1,0)$ with probability $(1-p)^2$;
- $(1,0,1,0) \rightarrow (0,1,1,0)$ with probability $p(1-p)$;
- $(1,0,1,0) \rightarrow (1,0,0,1)$ with probability $(1-p)p$;
- $(1,0,1,0) \rightarrow (0,1,0,1)$ with probability p^2 .

Next, consider the initial state $s_2 = (1,0,0,1)$. The possible transitions are:

- $(1,0,0,1) \rightarrow (1,0,0,1)$ with probability $1-p$;
- $(1,0,0,1) \rightarrow (0,1,0,1)$ with probability p .

Now consider the initial state $s_3 = (0,1,1,0)$. The possible transitions are:

- $(0,1,1,0) \rightarrow (0,1,1,0)$ with probability $1-p$;
- $(0,1,1,0) \rightarrow (0,1,0,1)$ with probability p .

Consider the initial state $s_4 = (0,1,0,1)$. The possible transitions are:

- $(0,1,0,1) \rightarrow (0,1,0,1)$ with probability $(1-p)^2$;
- $(0,1,0,1) \rightarrow (0,0,1,1)$ with probability $p(1-p)$;
- $(0,1,0,1) \rightarrow (1,1,0,0)$ with probability $(1-p)p$;
- $(0,1,0,1) \rightarrow (1,0,1,0)$ with probability p^2 .

Finally, consider the initial state $s_5 = (0,0,1,1)$. The possible transitions are:

- $(0,0,1,1) \rightarrow (0,0,1,1)$ with probability $1-p$;
- $(0,0,1,1) \rightarrow (1,0,1,0)$ with probability p .

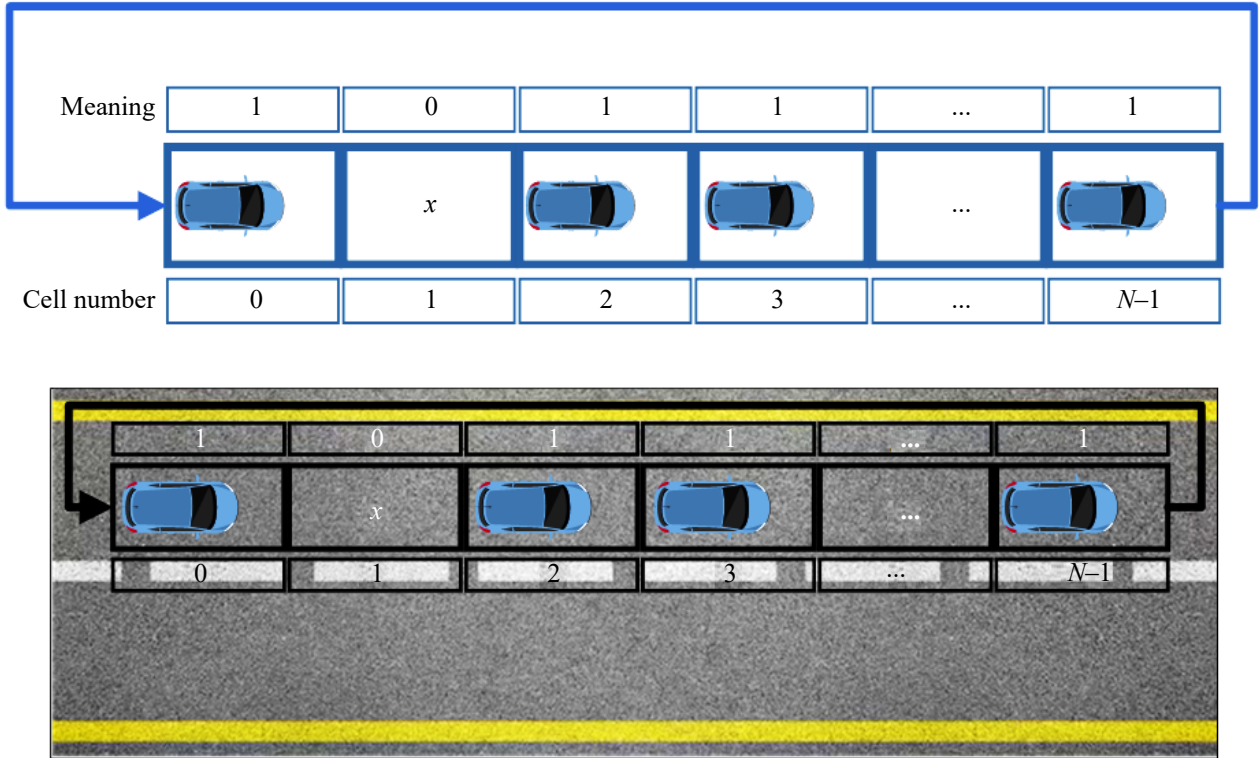


Fig. 2. Visualization of Cell Arrangement on a Ring

Thus, for $N = 4, M = 2$ we get the transition matrix $P_{6 \times 6}$:

$$P_{6 \times 6} = \begin{matrix} & \begin{matrix} s_0 & s_1 & s_2 & s_3 & s_4 & s_5 \end{matrix} \\ \begin{matrix} s_0 \\ s_1 \\ s_2 \\ s_3 \\ s_4 \\ s_5 \end{matrix} & \begin{pmatrix} 1-p & p & 0 & 0 & 0 & 0 \\ 0 & (1-p)^2 & p(1-p) & (1-p)p & p^2 & 0 \\ 0 & 0 & 1-p & 0 & p & 0 \\ 0 & 0 & 0 & 1-p & p & 0 \\ (1-p)p & p^2 & 0 & 0 & (1-p)^2 & p(1-p) \\ 0 & p & 0 & 0 & 0 & 1-p \end{pmatrix} \end{matrix}.$$

Let π_i be the stationary probability of state s_i ; p_{ij} be the probability of transitioning from state s_i to state s_j , $i, j = 0, 1, \dots, C_N^M - 1$.

The stationary probabilities of the states satisfy the system of equations:

$$\pi_i \sum_{j=0}^{C_N^M-1} p_{ij} = \sum_{j=0}^{C_N^M-1} \pi_{i+j} p_{ji}, \quad i, j = 0, \dots, C_N^M - 1, \quad (4)$$

$$\sum_{i=0}^{C_N^M-1} \pi_i = 1. \quad (5)$$

Let $k(s_i)$ be the number of “clusters” in the state s_i of the chain. A cluster is defined as a group of neighboring cells occupied by particles, separated from other clusters by free cells. If the chain is in state s_i , the number of particles that can move equals the number of clusters $k(s_i)$, $0 \leq k(s_i) \leq M$. There are $2^{k(i)}$ states into which the chain can transition from state s_i . Let the system, starting from state s_i , be able to reach state s_j , $1 < j < C_N^M - 1$ in one time step by moving $l(s_i, s_j)$ particles, $0 \leq l(s_i, s_j) \leq k(s_i)$. Then

$$p_{ij} = p^{l(s_i, s_j)} (1-p)^{k(s_i)-l(s_i, s_j)}, \quad l(s_i, s_i) = 0. \quad (6)$$

The probability that the system remains in state s_i at this step is the probability that none of the $k(s_i)$ particles will move, $p_{ii} = (1-p)^{k(s_i)}$, so

$$\sum_{j=0, j \neq i}^{C_N^M-1} p_{ij} = 1 - (1-p)^{k(s_i)}. \quad (7)$$

Let $S(k, N, M)$ be the number of states with $0 \leq k \leq M$ clusters for a given number of cells N and particles M .

Lemma 1. The following equality holds:

$$S(k, N, M) = v = \begin{cases} \frac{N}{k} C_{M-1}^{k-1} C_{N-M-1}^{k-1}, & k \leq \min(M, N-M), \\ 0, & k > \min(M, N-M). \end{cases} \quad (8)$$

Proof. Let $R(k, M)$ be the number of ways, taking into account the order, to represent the number M as a sum of k natural summands ($k \leq \min(M, N-M)$). This number is equal to the number of ways to choose $(k-1)$ elements from $M-1$:

$$R(k, M) = C_{M-1}^{k-1}. \quad (9)$$

Next, consider the number $A(k, N, M)$ of states in set A , that contain k clusters and where cell 1 is free, and cell 2 is occupied. This number is the product of the number of ways to distribute M particles among k clusters $R(k, M)$ multiplied by the number of ways to choose the lengths $(k-1)$ of the gaps (intervals between clusters) $R(k, N-M)$. Therefore, if $k \leq \min(M, N-M)$:

$$A(k, N, M) = R(k, M) R(k, N-M). \quad (10)$$

From Equations (9)–(10), we conclude that:

$$A(k, N, M) = C_{M-1}^{k-1} C_{N-M-1}^{k-1}. \quad (11)$$

Let $a = (i_1, \dots, i_N)$ be an arbitrary state from the set A . Define $b(a, d)$ the state corresponding to a by rotating it by d cells, i. e., $b(a, d) = (i_{N-d+1}, i_{N-d+2}, \dots, i_N, i_1, \dots, i_{N-d})$. Any state with k clusters coincides with $b(a, d)$ for k different ordered pairs (a, d) , where a is an element of the set A , and d is one of the numbers $0, 1, \dots, N-1$. The total number of such distinct pairs is $N \cdot A(k, N, M)$. Thus, we have:

$$S(k, N, M) = \frac{N}{k} A(k, N, M). \quad (12)$$

From equations (11) and (12), it follows that equation (9) holds. The lemma is proven.

Theorem 1. The system (4)–(5) has a solution

$$\pi_i = \frac{C}{(1-p)^{k(s_i)-1}}, \quad i = 0, \dots, C_N^M - 1, \quad (13)$$

where the constant is

$$C = \left(\sum_{k=1}^{\min(M, N-M)} \frac{N}{k} \cdot C_{M-1}^{k-1} C_{N-M-1}^{k-1} \frac{1}{(1-p)^{k-1}} \right)^{-1}. \quad (14)$$

Proof. Let's rewrite (4) in the form

$$\sum_{j=0}^{C_N^M-1} \pi_i p_{ij} = \sum_{j=0}^{C_N^M-1} \pi_j p_{ji}, i, j = 0, \dots, C_N^M - 1. \quad (15)$$

Let $F(s_i)$ be the set of states that can be reached from state s_i in one step; $F(s_i, l)$ is a subset of the set $F(s_i)$ containing states that can be reached from state s_i by transitioning l particles; $B(s_i)$ is a subset of $B(s_i)$, containing states that can transition into state s_i in one step; $B(s_i, l)$ is a subset of $B(s_i)$, containing states that can transition into state s_i by transitioning particles.

We have

$$p_{ij} = \begin{cases} 0, j \notin F(s_i), i, j = 0, \dots, C_N^M - 1, \\ p^l (1-p)^{k(s_i)-l}, j \in F(s_i, l), i, j = 0, \dots, C_N^M - 1, l = 1, \dots, k(s_i), \end{cases} \quad (16)$$

$$p_{ji} = \begin{cases} 0, j \notin B(s_i), i, j = 0, \dots, C_N^M - 1, \\ p^l (1-p)^{k(s_i)-l}, j \in B(s_i, l), i, j = 0, \dots, C_N^M - 1, l = 1, \dots, k(s_i). \end{cases} \quad (17)$$

Taking into account (16) and (17), we rewrite (15) in the form:

$$\sum_{l=1}^{k(s_i)} \text{card} F(s_i, l) \pi_i p^l (1-p)^{k(s_i)-l} = \sum_{l=1}^{k(s_i)} \sum_{j \in B(s_i, l)} \text{card} B(s_i, l) \pi_j p^l (1-p)^{k(s_i)-l}, i = 0, \dots, C_N^M - 1, \quad (18)$$

where “card” denotes the number of elements in the set A .

Substituting (13) into (18), we rewrite (18) in the form

$$\sum_{l=1}^{k(s_i)} \text{card} F(s_i, l) p^l (1-p)^{1-l} = \sum_{l=1}^{k(s_i)} \text{card} B(s_i, l) p^l (1-p)^{1-l}. \quad (19)$$

Due to the symmetry $\text{card} B(s_i, l) = \text{card} F(s_i, l), l = 1, \dots, k(s_i), i = 0, \dots, C_N^M - 1$.

Thus, the solution (13) satisfies the system (19), and consequently, also satisfies the system (4)–(5). The formula (14) for calculating the constant follows from (6), (13), and Lemma 1. Theorem 1 is proven.

Estimation of the Average Speed Based on Statistical Mechanics. In [11], the hypothesis is proposed that the average speed of particles follows the formula

$$v = \frac{1 - \sqrt{1 - 4\rho p(1-\rho)}}{2\rho}, \quad (20)$$

where $\rho = M / N$ is the particle flow density.

In [12], it is shown that for finite systems (finite N) correlations between vehicles and their mutual influence on movement cannot be fully accounted for by the formula (20). This is because in finite systems, there is a limited number of vehicles, and their motion can be more complex due to interactions.

In the limit as $N \rightarrow \infty$ (thermodynamic limit), the system becomes infinitely large, and correlations between vehicles become local. In this case, the formula (20) becomes exact, as the influence of boundary conditions and the finiteness of the system vanishes.

In [13], a rigorous proof of the formula for the average speed in a model based on asymmetric exclusion processes (Asymmetric Simple Exclusion Process – ASEP) on a ring is presented. This proof relies on the use of recursive formulas for the distribution function and hypergeometric functions, which allow obtaining an exact expression for the average speed as a function of density ρ and the probability p .

It is worth noting that for the creation of intelligent transport systems (ITS), short sections of highways are especially important, as monitoring traffic flows using cameras mounted on lighting masts or traffic lights has a limited coverage area of the lane.

Estimation of the Average Speed Based on the Stochastic Model. Since in real-world conditions the traffic analysis task is carried out on finite sections of highways, formulas based on probability theory can be used to estimate the average speed.

Let $H(t)$ be the average total number of particle movements over a time interval $(0, t)$. As follows from the ergodic theorem of Markov processes theory, the limit

$$v = \lim_{t \rightarrow \infty} \frac{H(t)}{Mt} \quad (21)$$

exists and does not depend on the initial configuration of the particles. The value v is called the particle speed.

Thus, the amount of movement of the system under consideration in one step is:

$$Q = \sum_{i=0}^{C_N^M-1} \pi_i k(s_i) p. \quad (22)$$

In [14], the following explicit formula for estimating the average speed of particles is obtained:

$$v = \frac{N}{M} \sum_{k=1}^{\min(M, N-M)} \frac{C}{(1-p)^{k-1}} \cdot C_{M-1}^{k-1} C_{N-M-1}^{k-1} \cdot p, \quad (23)$$

$$C = \left(\sum_{k=1}^{\min(M, N-M)} \frac{N}{k} \cdot C_{M-1}^{k-1} C_{N-M-1}^{k-1} \frac{1}{(1-p)^{k-1}} \right)^{-1},$$

where C_N^M is the number of combinations of N elements chosen from M .

The formula follows from the statement of Theorem 1, Lemma 1, and formula (22). In [15], an alternative derivation of formula (23) is given.

Estimation of Average Speed Based on Queueing Systems. A research team led by Doctor of Technical Sciences M.V. Yashina, which includes the author of this article, discovered that for calculating the average speed, a formula based on queueing systems, presented in [16], can be applied.

The average speed is equal to the limit of the ratio of recursive procedures:

$$v = \frac{r_2(N-M, M)}{r_1(N-M, M)}, \quad (24)$$

where $q = 1 - p$, and $r_1(N-M, M)$ and $r_2(N-M, M)$ are recursive procedures:

$$r_1(1, M) = \frac{M}{p}, \quad M \geq 1, \quad (25)$$

$$r_1(N-M, 1) = \left(\frac{q}{p} \right)^{N-M} \frac{1}{q}, \quad N-M \geq 1, \quad (26)$$

$$r_2(2, M) = \frac{q + M - 1}{p}, \quad M \geq 1, \quad (27)$$

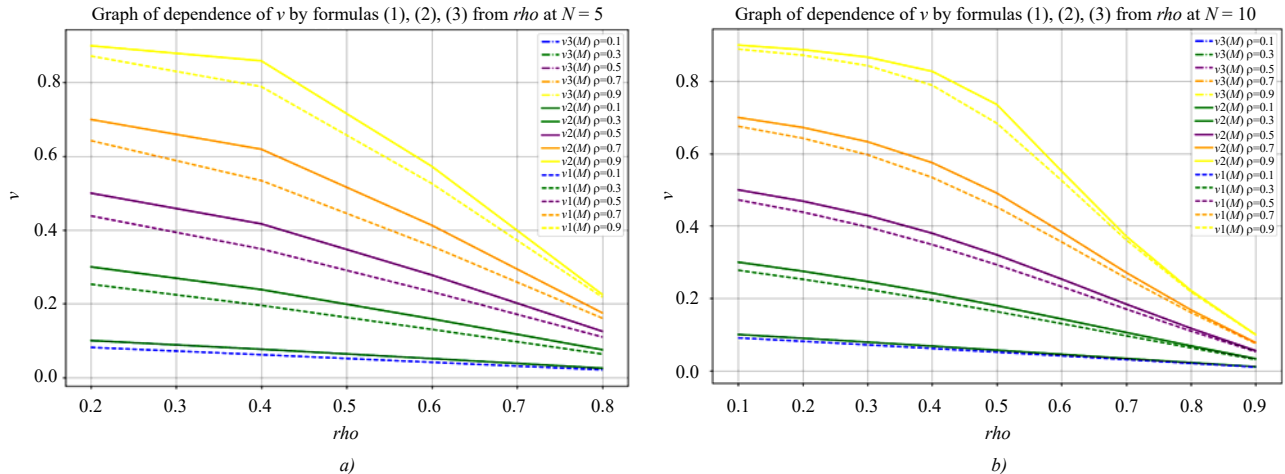
$$r_1(N-M, M) = r_1(N-M, M-1) + \frac{q}{p} r_2(N-M-1, M) + r_1(N-M-1, M-1), \quad N-M \geq 2, \quad M \geq 2, \quad (28)$$

$$r_2(N-M, M) = r_1(N-M-1, M-1) + \frac{q}{p} r_2(N-M-1, M), \quad N-M \geq 3, \quad M \geq 1. \quad (29)$$

Comparison of Average Speed Estimates from Proposed Methods. Let's compare the dependencies of the average speed estimate v on the density ρ for different values of the probability p using the formulas (20), (23), and (24) (Fig. 3).

The average speed estimate according to formulas (23) and (24) gives the same value, so the graphs overlap. From the graphs, it can be seen that for small values of N the average speed calculated using formula (20) differs from those using formulas (23) and (24). As N increases, the difference between the values decreases.

Figure 4 shows the dependency of the average speed v , calculated using formulas (20), (23), and (24), on the probability $p \in [0, 01, 0, 99]$ and the number of particles $M \in [1, N-1]$.



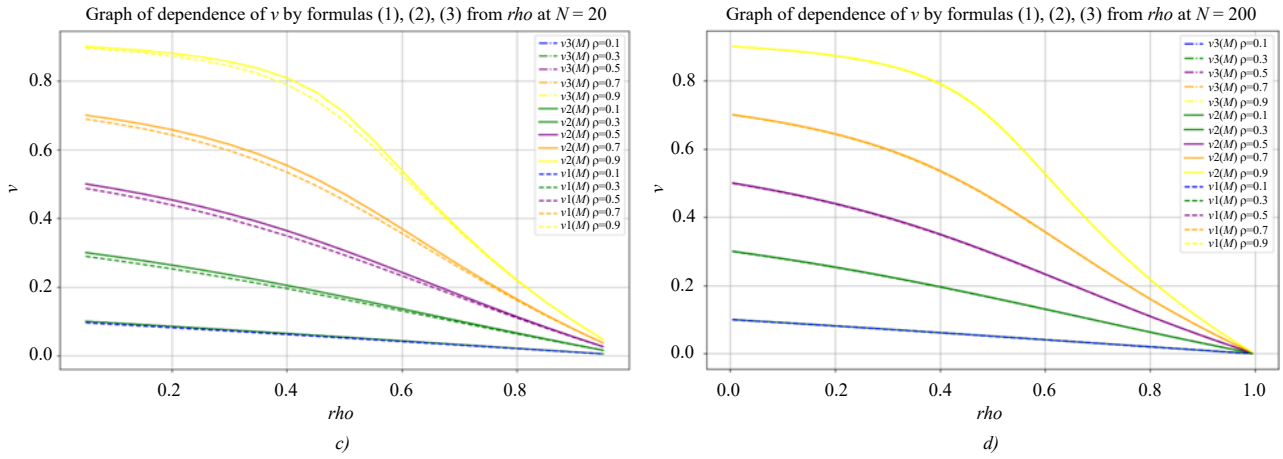


Fig. 3. Average speed estimate according to formulas (20), (23), and (24) for $a — N = 5$; $b — N = 10$; $c — N = 20$; $d — N = 200$

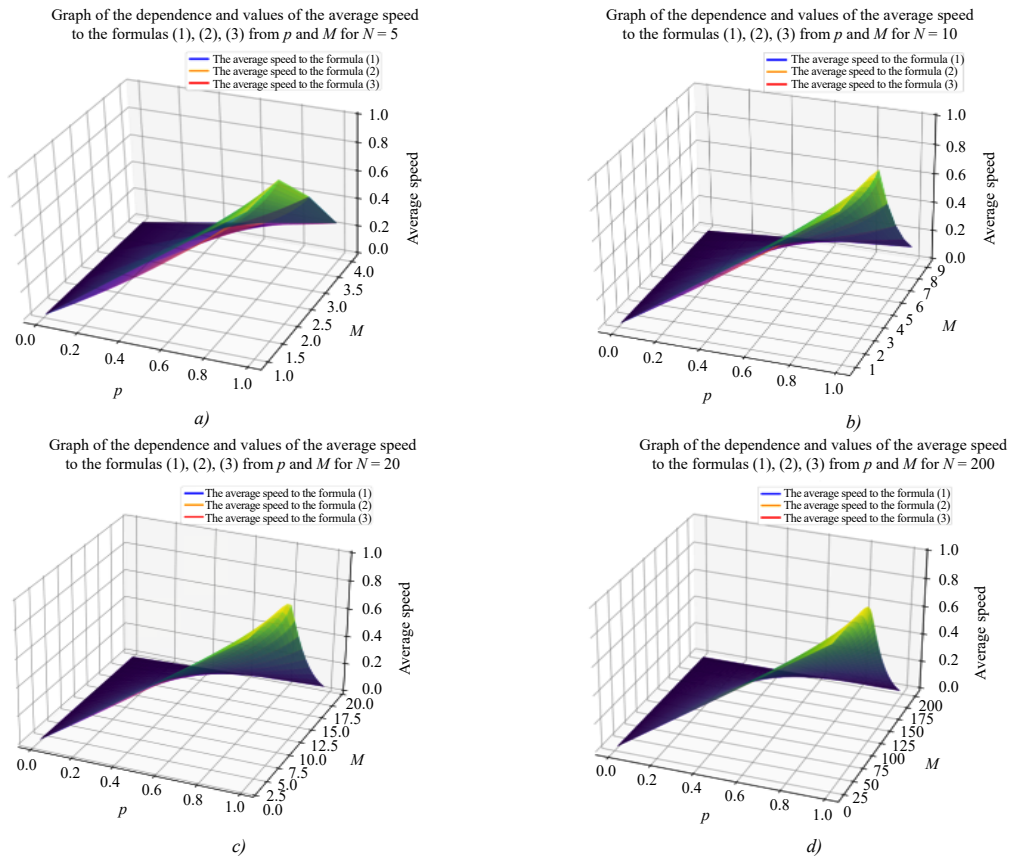


Fig. 4. Dependency of the average speed v , calculated using formulas (20), (23), and (24), on the probability $p \in [0,01, 0,99]$ and the number of particles $M \in [1, N-1]$ for $a — N = 5$; $b — N = 10$; $c — N = 20$; $d — N = 200$

From the graphs, we can evaluate the nature of the change in the average speed v , calculated using formulas (20), (23), and (24), with respect to the probability $p \in [0,01, 0,99]$ and the number of particles $M \in [1, N-1]$.

In Fig. 5, the difference in the values of the average speed v , calculated using formulas (20) and (23), is presented as a function of the probability $p \in [0,01, 0,99]$ and the number of particles $M \in [1, N-1]$. Formula (24) is not considered, as its values coincide with the average speed values calculated using formula (23).

From the graphs, the nature of the error growth is visible in two directions: when the probability p approaches 0.5, and at low particle density, as well as when the particle density ρ , approaches 1, and the particle density ρ , approaches 0.5. At low values of N the error is within the hundredths, while as N increases, the error decreases to thousandths.

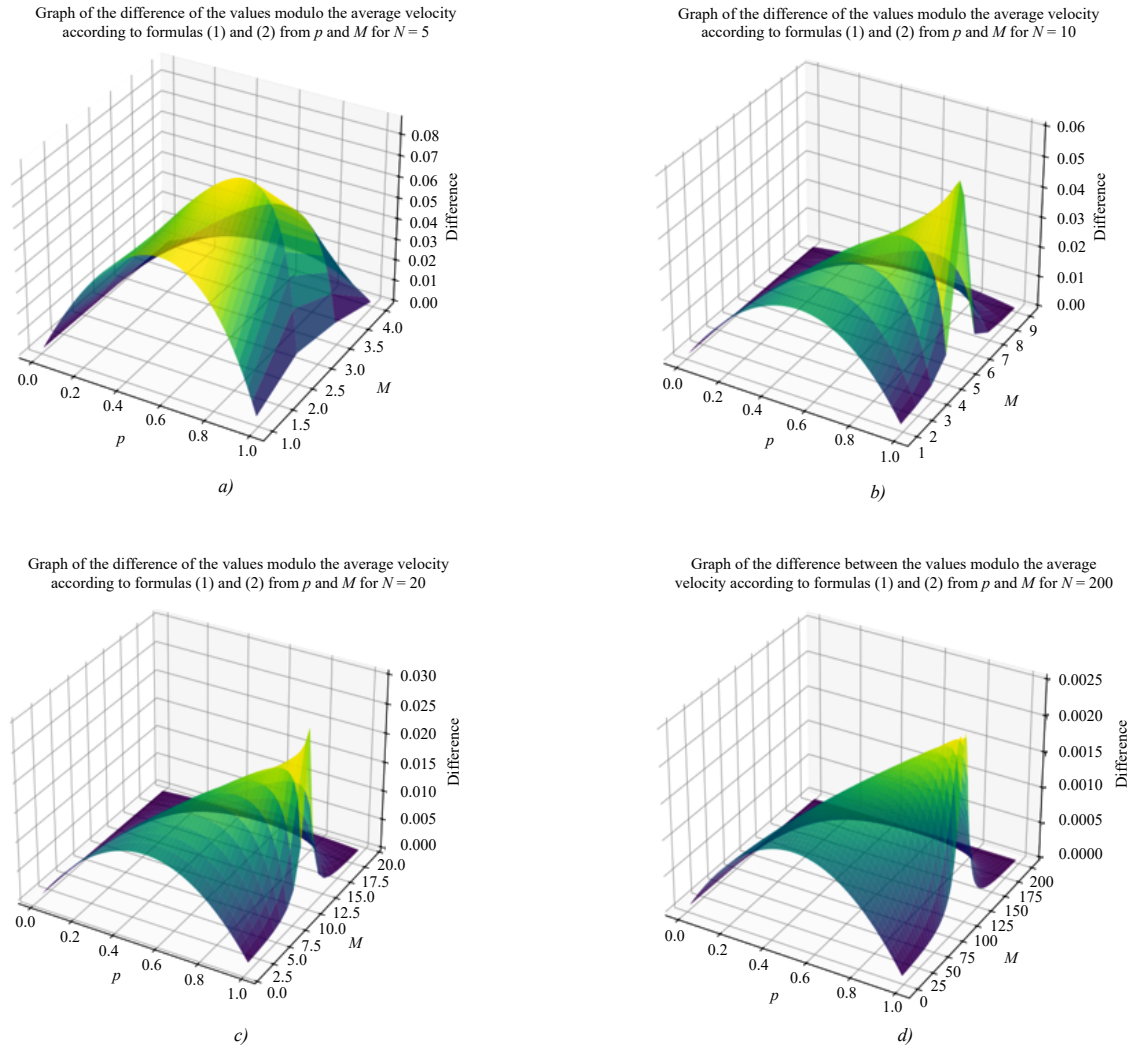


Fig. 5. The difference in the values of the average speed v , calculated using formulas (20) and (23), as a function of the probability $p \in [0.01, 0.99]$ and the number of particles $M \in [1, N-1]$ for a — $N = 10$, b — $N = 10$, c — $N = 20$, d — $N = 200$

Comparison of Average Speed Estimation with Computer Simulation. For the numerical estimation of average speed values calculated by formulas (20), (23), and (24), a computer simulation was implemented, modelling the movement of particles on a closed lattice. The user specifies the number of cells N the number of particles M , which are uniformly distributed across all cells, the probability of a particle moving to the next cell p if the next cell is empty, the number of iterations t , and the number of simulation runs T .

At each iteration, the instantaneous speed v is calculated as the number of particles that moved divided by the total number of particles, and the average speed is the sum of all instantaneous speeds over the number of iterations.

Upon completion of all runs, the average speed $\bar{v}$, is calculated as the sum of all average speeds at the end of each simulation run divided by the number of simulation runs. The average speeds were calculated for $t = 2000$, $T = 10$.

Tables 1–4 show the results of average speed calculations based on formulas (20) and (23) (formula (24) is not shown as the values computed by it are identical to those of formula (23)), as well as results from the computer simulation (sim) for different numbers of cells N .

Table 1

Average speed for $N = 5$

p/ρ	$\rho = 0.3$			$\rho = 0.5$			$\rho = 0.7$			$\rho = 0.9$		
	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim
$p = 0.1$	0.081	0.100	0.101	0.062	0.076	0.074	0.041	0.051	0.050	0.020	0.025	0.025
$p = 0.3$	0.253	0.300	0.302	0.195	0.238	0.237	0.130	0.159	0.159	0.063	0.075	0.074
$p = 0.5$	0.438	0.500	0.500	0.349	0.417	0.418	0.232	0.278	0.275	0.110	0.125	0.125
$p = 0.7$	0.643	0.700	0.700	0.534	0.619	0.622	0.356	0.413	0.415	0.161	0.175	0.173
$p = 0.9$	0.872	0.900	0.896	0.789	0.859	0.862	0.526	0.573	0.572	0.218	0.225	0.225

Table 2

Average speed for $N = 10$

p/ρ	$\rho = 0.1$			$\rho = 0.3$			$\rho = 0.5$			$\rho = 0.7$			$\rho = 0.9$		
	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim
$p = 0.1$	0.091	0.100	0.100	0.072	0.079	0.079	0.051	0.057	0.057	0.031	0.034	0.034	0.010	0.011	0.011
$p = 0.3$	0.278	0.300	0.299	0.225	0.247	0.243	0.163	0.180	0.182	0.097	0.106	0.104	0.031	0.033	0.032
$p = 0.5$	0.472	0.500	0.503	0.397	0.429	0.429	0.293	0.320	0.320	0.170	0.184	0.183	0.052	0.056	0.055
$p = 0.7$	0.676	0.700	0.705	0.597	0.633	0.632	0.452	0.491	0.490	0.256	0.271	0.269	0.075	0.078	0.078
$p = 0.9$	0.889	0.900	0.897	0.843	0.867	0.866	0.684	0.736	0.734	0.361	0.372	0.371	0.099	0.100	0.100

Table 3

Average speed for $N = 20$

p/ρ	$\rho = 0.1$			$\rho = 0.3$			$\rho = 0.5$			$\rho = 0.7$			$\rho = 0.9$		
	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim
$p = 0.1$	0.091	0.095	0.093	0.072	0.075	0.075	0.051	0.054	0.053	0.031	0.032	0.031	0.010	0.011	0.010
$p = 0.3$	0.278	0.289	0.285	0.225	0.235	0.232	0.163	0.171	0.171	0.097	0.101	0.101	0.031	0.032	0.031
$p = 0.5$	0.472	0.486	0.478	0.397	0.413	0.411	0.293	0.306	0.305	0.170	0.177	0.179	0.052	0.054	0.054
$p = 0.7$	0.676	0.688	0.691	0.597	0.615	0.615	0.452	0.471	0.472	0.256	0.264	0.263	0.075	0.076	0.076
$p = 0.9$	0.889	0.895	0.892	0.843	0.856	0.855	0.684	0.708	0.706	0.361	0.367	0.366	0.099	0.099	0.099

Table 4

Average speed for $N = 200$

p/ρ	$\rho = 0.1$			$\rho = 0.3$			$\rho = 0.5$			$\rho = 0.7$			$\rho = 0.9$		
	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim	(20)	(23)	sim
$p = 0.1$	0.091	0.091	0.091	0.072	0.072	0.070	0.051	0.052	0.051	0.031	0.031	0.030	0.010	0.010	0.010
$p = 0.3$	0.278	0.279	0.279	0.225	0.226	0.225	0.163	0.164	0.163	0.097	0.097	0.096	0.031	0.031	0.031
$p = 0.5$	0.472	0.474	0.474	0.397	0.399	0.399	0.293	0.294	0.294	0.170	0.171	0.170	0.052	0.053	0.052
$p = 0.7$	0.676	0.677	0.674	0.597	0.599	0.598	0.452	0.454	0.453	0.256	0.257	0.256	0.075	0.075	0.074
$p = 0.9$	0.889	0.890	0.887	0.843	0.845	0.842	0.684	0.686	0.685	0.361	0.362	0.361	0.099	0.099	0.074

From the tables, it is evident that the value of the average speed calculated using formulas (23) and (24) is closer to the simulation results for small values of N . In general, the error between the results for formulas (20) and (23)–(24) fully correlates with the results shown in Fig. 5. The obtained results also coincide with the results of the comparison between formulas (20) and (23) in [17, 18]. It can be concluded that as N increases, the results calculated using formula (20) increasingly approach the results obtained from formulas (23), (24), and computer simulation.

Computational Complexity Estimation of Average Speed Calculation Formulas. Let's estimate the computational complexity of calculating the average speed using formula (20). Since formula (20) does not contain loops or recursions, and its computation depends only on basic operations and constants, its computational complexity is constant — $O(1)$.

Now, let's estimate the computational complexity of calculating the average speed using formula (23). The sum iterates over all values of k from 1 to $\min(M, N - M)$. Let $K = \min(M, N - M)$ be the number of terms. Then the sum has K terms.

For each term in the sum, we need to compute C_{M-1}^{k-1} , which requires $O(k)$ operations, and C_{N-M-1}^{k-1} , which also requires $O(k)$ operations. Computing the remaining values and multiplying all these values together requires $O(1)$ operations. Thus, the overall computational complexity for one term in the sum is $O(k)$. Since the sum has K terms, the total computational complexity for calculating the entire sum is $O(\sum_{k=1}^K k) = O(K^2)$.

The total computational complexity of formula (23) is determined by the complexity of computing the normalization coefficient C and the expression for v . Since both of these computations have a complexity of $O(K^2)$, the total complexity is also $O(K^2)$, where $K = \min(M, N - M)$.

Let's estimate the computational complexity of calculating the average speed using formula (24). Both functions r_1 and r_2 depend on two parameters: $N - M$ and M . Therefore, the total number of unique states (or nodes in the recursion tree) equals the number of possible combinations $(M, N - M)$, where $N - M$ can take values from 1 to $N - 1$, and M can take values from 1 to $\min(M, N - M)$. Consequently, the total number of states is $O(N^2)$.

The recursion depth for each function is determined by the maximum value of $N - M$ or M . In the worst case, the recursion depth is $O(N)$. At each recursion level, simple arithmetic operations are performed, requiring $O(1)$ operations. Thus, the total number of operations is proportional to the number of unique states.

Considering the above factors, the total computational complexity of formula (24) is $O(N^2)$. This is because each state $(M, N - M)$ is computed exactly once, and the total number of states is $O(N^2)$. Therefore, formula (20) is the most computationally efficient. Formulas (23) and (24) have identical computational complexities.

Discussion and Conclusion. The paper presents an estimation of the limiting speed of one-way traffic flow with high computational efficiency. The main conclusions are as follows:

1. For small values of N , which are most representative of real highway sections captured by cameras, formulas (23) and (24) are the most suitable for estimating the average speed v . These formulas provide equivalent results that closely match the outcomes of computer simulations..

2. For estimating the average speed v for large N computer calculations with formula (23) are constrained by memory usage due to the need to calculate large factorials. When $N < 310$ exceeds a certain threshold, memory overflow occurs on the used computer system $\rho \approx \frac{1}{2}$. In such cases, formula (24) can be used as an alternative.

3. For estimating the average speed v for large N computer calculations with formula (24) are constrained by the maximum recursion depth for the values of r_1 and r_2 . As a result, even when using dynamic programming methods, the computation may take longer than when using formula (23) and may lead to a stack overflow.

4. From the obtained results, it is clear that as N increases, the estimate of the average speed using formula (20) will approach the values computed using formulas (23) and (24), while requiring significantly fewer computational resources. Thus, formula (23), yielding an equivalent result to formula (24) for $N \rightarrow \infty$, can be applied in problems related to queuing systems.

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