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UDC 534.11

https://doi.org/10.23947/2587-8999-2025-9-2-34-43

Original Empirical Research

Study of the Influence of Boundary Motion on the Oscillatory and Resonance Properties of Mechanical Systems with Variable Length

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Abstract

Introduction. The widespread use of technical systems with moving boundaries necessitates the development of mathematical modelling methods and algorithmic software for their analysis. This paper presents a systematic review of studies examining the oscillatory and resonance properties of mechanical systems with moving boundaries, such as hoisting cables, flexible transmission mechanisms, strings, rods, beams with variable length, and others.

Materials and Methods. A problem statement is formulated, and numerical methods are developed for solving nonlinear problems that describe wave processes and the resonance properties of systems with moving boundaries.

Results. An analysis is conducted on wave reflection from moving boundaries, including changes in their energy and frequency. It is shown that the energy of the system increases when the boundary moves toward the waves and decreases when moving in the same direction as the waves. Criteria are obtained to determine the conditions under which the boundary motion must be considered for accurate calculation of oscillation amplitudes. Numerical results demonstrate the influence of boundary speed and damping on the system dynamics.

Discussion and Conclusion. The findings have practical significance for the design and operation of mechanical systems with variable geometry. The results make it possible to prevent large-amplitude oscillations in mechanical objects with moving boundaries at the design stage. These problems have not been sufficiently studied and require further research.

Keywords: resonance properties, vibrations of systems with moving boundaries, wave processes, damping, vibration amplitude

For Citation. Semenov A.L., Litvinov V.L., Shamolin M.V. Study of the Influence of Boundary Motion on the Oscillatory and Resonance Properties of Mechanical Systems with Variable Length. *Computational Mathematics and Information Technologies*. 2025;9(2):34–43. https://doi.org/10.23947/2587-8999-2025-9-2-34-43

Оригинальное эмпирическое исследование

Исследование влияния движения границ на колебательные и резонансные свойства механических систем переменной длины

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Аннотация

Введение. Широкое распространение в технике объектов с движущимися границами обусловливает необходимость развития методов математического моделирования и создания алгоритмического программного обеспечения для соответствующего анализа. Настоящая работа представляет собой систематизированный обзор материалов, в которых исследуются колебательные и резонансные свойства механических систем с движущимися границами, таких как канаты подъемных устройств, гибкие передаточные механизмы, струны, стержни, балки переменной длины и т. д.

Материалы и методы. Сформулирована постановка и разработаны численные методы решения нелинейных задач, описывающих волновые процессы и резонансные свойства объектов с движущимися границами.
Результаты исследования. Проведен анализ отражения волн от движущихся границ, включая изменение их энергии и частоты. Показано, что энергия системы возрастает при движении границы навстречу волнам и убывает при совпадении направлений. Получены критерии, определяющие условия, при которых необходимо учитывать движение границ для корректного расчета амплитуд колебаний. Численные результаты демонстрируют влияние скорости движения границ и демпфирования на динамику системы.
Обсуждение и заключение. Результаты работы имеют практическое значение для проектирования и эксплуатации механических систем с переменной геометрией. Приведенные результаты позволяют на стадии проектирования предотвратить возможность возникновения колебаний большой амплитуды в механических объектах с движущимися границами. Данные задачи мало изучены и требуют дальнейшего исследования.

Ключевые слова: резонансные свойства, колебания систем с движущимися границами, волновые процессы, демпфирование, амплитуда колебаний

Для цитирования. Семенов А.Л., Литвинов В.Л., Шамолин М.В. Исследование влияния движения границ на колебательные и резонансные свойства механических систем переменной длины. *Computational Mathematics and Information Technologies*. 2025;9(2):34–43. https://doi.org/10.23947/2587-8999-2025-9-2-34-43

Introduction. In the field of elastic system dynamics, particular practical interest is drawn to problems involving vibrations of structures whose geometric parameters change over time. Typical examples of such systems include hoisting ropes [1–8], flexible transmission elements [4, 6, 9], drilling rigs [10], and others. Numerous studies on the dynamics of hoisting ropes have revealed the need to develop new approaches to analyzing the behavior of one-dimensional objects with variable geometric characteristics.

Similar problems involving moving boundaries also arise in the context of heat transfer, thermal conductivity, and diffusion equations (notably, the Stefan problem). Such issues have been addressed in the works of L.A. Uvarova [11], V.A. Kudinov [12], and other researchers.

A related class of problems—devoted to constructing two- and three-dimensional mathematical models of marine and coastal systems, shallow water bodies, wave hydrodynamics and geophysics, and the correctness of problem formulations described by elliptic-type equations—has been investigated by A.I. Sukhinov and his students [13, 14]. These authors study the development and analysis of two-dimensional-one-dimensional splitting schemes and methods for solving grid-based diffusion-convection-reaction problems, which form the basis for efficient parallel algorithms.

The results of A.I. Sukhinov, A.M. Atayan, A.V. Nikitina, A.E. Chistyakov, V.V. Sidoryakina [15], and others form the foundation for studying forecasting problems of adverse and hazardous phenomena, including wave processes at boundaries in natural and man-made systems; mass transfer across moving boundaries such as storm surges, coastal flooding, and the formation of hypoxic zones in marine and coastal systems using precision models; as well as for remote sensing and artificial intelligence applications. These authors have examined the existence and uniqueness of solutions to linearized initial-boundary value problems for the developed models.

The problem of vibrations in systems with moving boundaries is related to obtaining solutions of systems of partial differential equations in time-varying domains, as well as integro-differential equations with time-dependent integration limits and kernels. It involves introducing the concepts of "eigenvalues" and "eigenfunctions" for variable-length objects and developing a general framework for studying boundary value problems of this class based on the synthesis of integral equation theory and asymptotic methods. Characteristic model boundary value problems are solved in the context of the dynamics of hoisting ropes, beams, rods, and strings with variable length, and their resonance properties are analyzed. These problems have not been sufficiently studied. Traditional methods of mathematical physics are mainly limited to problems with fixed boundaries.

The difficulties encountered in formulating and solving such problems stem from the fact that, to date, no sufficiently general approach exists for analyzing the dynamic behavior of such systems. Existing results are limited to qualitative descriptions of dynamic phenomena, while little attention has been given in the literature to obtaining quantitative characteristics with practical value.

The theoretical significance of this study lies in the development and investigation of new mathematical models describing the vibrations of objects with moving boundaries in the form of partial differential equations.

The practical significance consists in the generalization of modelling techniques and numerical analysis of the resonance properties of objects described by boundary value problems with moving boundaries. The emergence of large-amplitude oscillations in such systems is often unacceptable, making resonance analysis a key focus.

From a mathematical perspective, these problems require solving hyperbolic-type equations in domains with moving boundaries. The considerable challenges in describing such systems justify the predominant use of approximate analytical methods. Among the analytical approaches, the most effective are those based on special variable transformations [16, 17], as well as methods employing the principle of superposition of counter-propagating wave processes [18]. Of particular interest is the approach proposed in [19], which involves using complex variable substitutions to reduce the original problem to the analysis of the Laplace equation.

However, the capabilities of exact analytical methods are significantly limited [1–3, 20–21]. Among approximate methods, special attention should be given to the Kantorovich-Galerkin method [10, 22], as well as approaches based on constructing solutions to integro-differential equations [23].

In systems with moving boundaries, two types of resonance phenomena are observed [4]: steady-state resonance and passage through resonance.

If a system with time-varying dimensions is subjected to an external force whose variation is synchronized with the changing natural frequency, the phenomenon of continuous amplitude growth is referred to as steady-state (or generalized) resonance. Passage through resonance refers to the sharp increase in amplitude over a finite time interval, during which the instantaneous natural frequency of one of the modes coincides with the excitation frequency.

Passage through resonance occurs over a limited time interval and typically does not reach the amplitude levels characteristic of steady-state resonance. However, when damping is high and boundary motion is slow, the amplitude values of both resonance types are close. In such situations, to estimate the vibration amplitude during passage through resonance, it is sufficient to fix the boundaries at the resonance point and compute the amplitude of the steady-state oscillations, which will approximate the maximum amplitude observed during the passage through resonance. Thus, the amplitude in the fixed-boundary case provides an upper bound estimate for the desired quantity.

Consequently, there is a need to expand the range of problems related to modelling vibrations in systems with moving boundaries and to develop new solution methods and corresponding software tools. This need constitutes the main objective of the present work. The study explores the patterns of wave reflection from moving boundaries in systems whose vibrations are described by the wave equation, as well as the interaction of longitudinal waves with moving boundaries. The influence of damping forces on vibration amplitudes during resonance passage in systems with moving boundaries is analyzed. Inequalities are derived that define the domains in which boundary motion must be accounted for. The paper presents a systematized review of materials previously presented by the authors at scientific conferences [24–26], which examine the vibrational and resonance properties of mechanical systems with moving boundaries.

Materials and Methods

Investigation of wave reflection patterns from moving boundaries. Let the oscillatory processes of the system be described by the wave equation:

$$U_{tt}(x,t) - a^2 U_{xx}(x,t) = 0. \quad (1)$$

Here $U(x,t)$ is the function representing longitudinal or transverse displacement of the object from the equilibrium position; t is the time; x is the spatial coordinate.

The oscillating object (string, rod) is unbounded on one side, while the other boundary moves according to a law $x = l(t)$. A sinusoidal wave $g(x + at)$ is incident on the moving boundary, where

$$g(z) = A\sin(wz + \gamma), \quad (2)$$

and a reflected wave $q(x - at)$ emerges from the boundary.

The task is to determine the change in energy of the reflected wave compared to the incident wave under uniform and periodic boundary motion. The solution to equation (1) is written in the form:

$$U(x,t) = g(x+at) + q(x-at). \quad (3)$$

The energy of the segment of the object ($x \in [a;b]$) is given by the formula:

$$W = \frac{1}{2}\rho\int_a^b (a^2 U_x^2(x,t) + U_t^2(x,t))dx, \quad (4)$$

where ρ is the linear mass density of the object.

Substituting expression (3) into (4), we obtain:

$$W = \frac{1}{2}\rho a^2 \int_a^b ((g'(x+at))^2 + (q'(x-at))^2)dx.$$

Thus, the system's energy consists of two parts — the energy of the incident wave and the energy of the reflected wave:

$$W_{пад.} = \frac{1}{2}\rho a^2 \int_a^b (g'(x+at))^2 dx, \quad (5)$$

$$W_{отр.} = \frac{1}{2}\rho a^2 \int_a^b (q'(x-at))^2 dx. \quad (6)$$

We will also use the dimensionless characteristic

$$W_0 = \frac{W_{отр.}}{W_{пад.}} \tag{7}$$

and the dimensionless variables:

$$U(x,t) = AY(\xi,\tau),\ \ \tau = wat,\ \ \xi = wx,\ \ p = wz,\ \ q(z) = AQ(p),\ \ g(z) = AG(p).$$

Then expressions (1)–(3), (5), and (6) will take the following form:

$$Y_{\tau\tau}(\xi,\tau) - Y_{\xi\xi}(\xi,\tau) = 0, \tag{8}$$

$$G(p) = \sin(p+\gamma), \tag{9}$$

$$Y(\xi,\tau) = G(\xi+\tau) + Q(\xi-\tau), \tag{10}$$

$$W_{пад.} = C\int_{a_0}^{b_0} (G'(\xi+\tau))^2 d\xi,\ \ W_{отр.} = C\int_{a_0}^{b_0} (Q'(\xi-\tau))^2 d\xi,$$

where $C = \frac{1}{2}\rho a^2 A^2 w,\ \ a_0 = wa,\ \ b_0 = wb.$

Consider the boundary condition at the moving boundary of the form:

$$U(l(t),t) = 0 \tag{11}$$

for uniform boundary motion $l(t)=Vt$.

In dimensionless variables, the boundary condition will have the form:

$$Y(L(\tau),\tau) = 0, \tag{12}$$

where

$$L(\tau) = \alpha\tau,\ \ \alpha = V/a\ \ (\alpha < 1). \tag{13}$$

Substitute the solution (10) into the boundary condition (12). As a result, we obtain:

$$G(L(\tau)+\tau) + Q(L(\tau)-\tau) = 0. \tag{14}$$

Let us denote this equation $P = (L(\tau) - \tau)$ and find from it the relation for τ: $\tau = \varphi(P)$:

Express: $L(\tau) + \tau = P + 2\varphi(P)$. When the boundary moves according to the law $\varphi(P) = \frac{z}{\alpha - 1}$ equation (14) becomes: $Q(P) = -G\left(\frac{\alpha+1}{\alpha-1}P\right)$.

Given that the incident wave is defined by expression (9), the reflected wave will have the form:

$$Q(P) = -\sin\left(-\frac{1+\alpha}{1-\alpha}P + \gamma\right). \tag{15}$$

Analysis of equation (15) shows that the amplitude of the wave does not change upon reflection from a moving boundary, while the frequency changes in accordance with the Doppler effect by a factor of $\frac{1+\alpha}{1-\alpha}$. When the boundary moves toward the wave, the frequency increases ($\alpha > 0$), when the boundary moves in the same direction as the wave, the frequency decreases ($\alpha < 0$).

Let us now calculate the energy change of a single incident wave upon reflection.

The wavelength of the incident wave (from equation (9)) is 2π. The wavelength of the reflected wave is $\frac{1-\alpha}{1+\alpha}2\pi$, therefore the ratio of the energies becomes

$$W_{пад.} = C\int_0^{2\pi} \cos^2(P+\gamma)dP = C\pi, \tag{16}$$

$$W_{отр.} = C\int_0^{2\pi\frac{1-\alpha}{1+\alpha}} \left(\frac{1+\alpha}{1-\alpha}\right)^2 \cos^2\left(\frac{1+\alpha}{1-\alpha}P + \gamma\right)dP = C\pi\left(\frac{1+\alpha}{1-\alpha}\right),$$

$$W_0 = \frac{W_{отр.}}{W_{пад.}} = \frac{1+\alpha}{1-\alpha}.$$

The energy of the system increases when the boundary moves toward the wave. The energy decreases when the boundary moves in the same direction as the wave.

Now consider periodic boundary motion:

$$l(t) = B\sin(\omega t). \tag{17}$$

Let us synchronize the motion of the boundary with the incident waves in such a way that during the time it takes for one wave to arrive ($T = 2\pi/wa$) the boundary completes an integer number of oscillations, denoted by n. In this case

$$\omega = wan. \tag{18}$$

Expression (17) in dimensionless variables $L(\tau) = wl(t), \tau = wat,\ \xi = wx$ taking into account (18) takes the form:

$$L(\tau) = \beta \sin(n\tau), \tag{19}$$

where $\beta = Bw$.

For subsonic boundary motion, the condition $(|L'(\tau)| < 1)$ must be satisfied $\beta n < 1$. Substituting (19) into the boundary condition (15) for the reflected wave yields:

$$Q(P) = -\sin(P + 2\varphi(P) + \gamma). \tag{20}$$

The function $\varphi(P)$ is defined implicitly and determined by the equation:

$$\beta \sin(n\varphi(P)) - \varphi(P) = P. \tag{21}$$

To determine the energy of the reflected wave, we find from (21) $\varphi'(P)$ and from (20) $Q'(P)$:

$$\varphi'(P) = 1/(\beta n \cos(\varphi(P)) - 1), \tag{22}$$

$$Q'(P) = -(1 + 2\varphi'(P))\cos(P + 2\varphi(P) + \gamma). \tag{23}$$

The energy of the incident wave is defined by expression (16).

Taking into account (22) and (23), the energy of the reflected wave is given by:

$$W_{omp.} = C\int_0^{2\pi} (\frac{\beta n \cos\varphi + 1}{\beta n \cos\varphi - 1} \cos(P + 2\varphi(P) + \gamma)^2 dP. \tag{24}$$

Results. We analyze expression (24) using the developed software package [27] to find its maximum with respect to β and γ for different values of n.

As a result of the numerical analysis, it was established that for any values of β the maximum energy of the reflected wave is achieved at $n = 2$ when $\gamma = \pi / 2$. For other values of n the maximum is reached at different values of $\gamma = 0$. It was also found that the function $W_0(\gamma)$ is periodic with period π for any values of n.

The dependence of W_0 on β on γ for $n = 2$, is presented in Table 1.

Table 1

Dependence of W_0 on β and γ for $n = 2$

γ \ β	0.000	0.045	0.090	0.135	0.180	0.225	0.270	0.315	0.360	0.405
0.00	1.000	0.955	0.989	1.096	1.280	1.559	1.973	2.608	3.658	5.661
0.31	1.000	0.969	1.015	1.132	1.325	1.615	2.043	2.695	3.764	5.773
0.63	1.000	1.008	1.082	1.224	1.443	1.762	2.226	2.924	4.047	6.089
0.94	1.000	1.055	1.165	1.338	1.588	1.943	2.453	3.208	4.398	6.488
1.26	1.000	1.093	1.232	1.430	1.706	2.090	2.636	3.438	4.683	6.818
1.57	1.000	1.108	1.258	1.465	1.750	2.146	2.706	3.526	4.794	6.952
1.88	1.000	1.093	1.232	1.430	1.706	2.090	2.637	3.439	4.688	6.840
2.20	1.000	1.055	1.165	1.338	1.588	1.944	2.453	3.210	4.405	6.524
2.51	1.000	1.008	1.082	1.224	1.443	1.762	2.227	2.926	4.054	6.125
2.83	1.000	0.969	1.015	1.132	1.325	1.616	2.044	2.696	3.769	5.795
3.13	1.000	0.955	0.989	1.096	1.280	1.559	1.973	2.608	3.658	5.661

Features of Longitudinal Wave Interaction with a Moving Boundary. Let us consider the propagation of longitudinal waves in a semi-infinite rod, where the left boundary moves between two rollers rotating with a circumferential speed v and simultaneously translating along the x-axis with the same speed.

Until now, in the formulation of similar problems, the fact that deformed sections of the rod pass through the boundary has been neglected, and the boundary condition in the absence of slip was written as:

$$U_t(l(t), t) = 0;\ l(t) = vt.$$

In cases where the deformations are significant, this can lead to substantial errors.

Let $U(x,t)$ be the longitudinal displacement of the cross-section of the rod at coordinate x at time t which satisfies the wave equation (1). If deformations are taken into account, the boundary condition remains the same:

$$U_t(l(t),t) = 0, \tag{25}$$

however, the law of boundary motion becomes coupled with $U(x,t)$ by the relation:

$$l'(t) = \nu / (1 + U_x(l(t),t)). \tag{26}$$

This dependence of the boundary's motion law on the oscillatory process makes the problem nonlinear. Problems of this kind are currently poorly studied. A similar problem was first considered in [21].

We study the influence of the deformation magnitude and the boundary's speed on the process of reflection of a harmonic wave:

$$\varphi(x + at) = A\sin\omega(x + at) \tag{27}$$

from a moving boundary.

Let us introduce the following dimensionless variables into the problem (1), (25)–(27):

$$U(x,t) = AV(\xi,\tau),\ l(t) = L(\tau)/\omega,$$
$$\xi = \omega x,\ \tau = a\omega t,\ \varphi(x + at) = Ag(\xi + \tau).$$

As a result, we obtain:

$$V_{\tau\tau}(\xi,\tau) - V_{\xi\xi}(\xi,\tau) = 0,\ V_\tau(L(\tau),\tau) = 0,$$
$$L'(\tau) = \varepsilon / (1 + \alpha V_\xi(L(\tau),\tau)),\ g(\xi + \tau) = \sin(\xi + \tau),\ \varepsilon = \nu / a,\ \alpha = A\omega.$$

We seek the solution in the form:

$$V(\xi,\tau) = \sin(\xi + \tau) + G(\xi - \tau).$$

As a result, to determine the functions G and L we obtain the following system:

$$L'(\tau) = \varepsilon / (1 + 2\alpha\cos(L(\tau) + \tau)),\ G'(L(\tau) - \tau) = \cos(L(\tau) + \tau).$$

From the second equation of the system, it follows that the amplitude of the deformation waves does not change. A comparison of the system's solution (the system was solved numerically using the developed software package [27]) with the solution that does not account for the change in $L(\tau)$ due to deformation, namely:

$$L(\tau) = \varepsilon\tau,\ G'(z) = \cos((\varepsilon + 1)z / (\varepsilon - 1)),\ z = \tau(\varepsilon - 1),$$

shows that there is a constant phase shift over time between the solutions. The wavelength in the first case is shorter. The phase shift per unit time, depending on the parameters ε and α deformation magnitude, is presented in Table 2.

Table 2

Phase shift per unit time depending on ε and deformation magnitude α

ε \ α	0.1	0.2	0.3	0.4
0.1	0.004	0.008	0.034	0.109
0.3	0.019	0.045	0.120	–
0.5	0.032	0.077	–	–
0.7	0.057	–	–	–

At certain moments in time, the boundary may move faster than the speed of sound $(L'(\tau) > 1)$. In such cases, the formulated problem becomes incorrect. The inequality $\varepsilon + 2\alpha < 1$ defines the admissible domain. In the cells of the table where this inequality is not satisfied, a dash (—) is used.

Analysis of the Influence of Boundary Motion in the Study of Resonance Properties of Systems with Damping. To answer the question of when it is necessary to take boundary motion into account, let us consider the process of passing through resonance in a system with damping.

In works [10, 28], the resonance properties of two variable-length systems under the influence of damping forces were studied. The expressions for the oscillation amplitude obtained therein take the form:

$$A_n^2(\tau) = E_n^2(\varepsilon\tau)e^{-2\alpha_0(\varepsilon_1\tau)\tau}\{[\int_0^\tau F_n(\varepsilon\zeta)e^{\alpha_0(\varepsilon_1\zeta)\zeta}\sin\Phi_n(\zeta)d\zeta]^2 + [\int_0^\tau F_n(\varepsilon\zeta)e^{\alpha_0(\varepsilon_1\zeta)\zeta}\cos\Phi_n(\zeta)d\zeta]^2\}, \tag{28}$$

where $\alpha_0(\varepsilon_1\tau), E_n(\varepsilon\tau), F_n(\varepsilon\zeta), \Phi_n(\zeta)$ are certain functions.

Omitting some mathematical derivations, we obtain the expression for (28) in the form:

$$A_{0n}^2(\tau_1,\tau_2) = A^2 \frac{2}{|\nu|} A_n^2(z_1,z_2),$$

where

$$A_n^2(z_1,z_2) = e^{-2\alpha Z_2}[I_s^2(z_1,z_2) + I_c^2(z_1,z_2)], \quad (29)$$

$$I_s(z_1,z_2) = \int_{z_1}^{z_2} e^{\alpha z}\sin(\pm z^2)dz,\ \ I_c(z_1,z_2) = \int_{z_1}^{z_2} e^{\alpha z}\cos(\pm z^2)dz, \quad (30)$$

$$\alpha = \alpha_0\sqrt{2/|\nu|},\ z_i = (\nu\tau_i + \omega_0)/\sqrt{2|\nu|},\ i = \overline{1,2}.$$

Here v is a parameter characterizing the speed of resonance passage; α_0 is a coefficient characterizing damping in the system; A is a constant value; τ_1,τ_2 are the boundaries of the resonance region.

Let us analyze expression (29) for its maximum in the vicinity of the point $z_0 = 0$.

As a result of the numerical solution of (29) using the developed software package [27], Table 3 was obtained.

Table 3

Results of the Numerical Solution of Expression (29) for the Maximum

α	0.00	0.10	0.30	0.50	0.70	1.00	1.30	2.00	3.00	7.00
z_1	–1.56	–1.54	–1.49	–1.49	–1.48	–1.48	–1.47	–1.46	–1.35	–1.29
z_2	1.56	1.45	1.30	1.25	1.20	1.15	1.10	1.00	0.70	0.40
$A_n(\alpha)$	2.37	2.06	1.60	1.29	1.07	0.84	0.68	0.47	0.33	0.144

The maximum oscillation amplitude that arises when the boundaries stop at the resonance point is determined by expression (28) at $v = 0$. Performing the calculations, we obtain:

$$A_{0n}^{\max} = A/\alpha_0. \quad (31)$$

When $v \neq 0$ the amplitude is determined by the expression:

$$A_{0n} = A\sqrt{\frac{2}{|\nu|}} A_n\left(\alpha_0\sqrt{\frac{2}{|\nu|}}\right), \quad (32)$$

where the value of the function A_n is taken from Table 3.

The boundary motion should be taken into account when the relative amplitude error

$$\Delta = \frac{A_{0n}^{\max} - A_{0n}}{A_{0n}} \quad (33)$$

is large.

Using the data from the table, it is easy to establish that the error Δ exceeds the value of 0.05 when

$$\alpha_0\sqrt{\frac{2}{|\nu|}} < 2{,}164. \quad (34)$$

Inequality (34) defines the region in the parameter space α_0, v, where boundary motion must be considered. Substituting into (34) and performing the transformations, we obtain the following inequality, which defines the region where boundary motion must be considered:

$$\Delta_A > 3{,}8\sqrt{\gamma\Delta_\ell},$$

where $\Delta_A = 2\pi\alpha_0/\omega_0$ is the relative change in amplitude over one free oscillation; $\Delta_\ell = 2\pi|\upsilon|/\omega_0\ell_0$ is the relative change in length over one free oscillation.

Discussion and Conclusion. The patterns of wave reflection from moving boundaries in systems whose oscillations are described by the wave equation have been investigated. An expression has been obtained for the change in the energy of the reflected wave relative to the incident wave in the case of uniform and periodic boundary motion. It has been established that the system's energy increases when the boundary moves towards the wave and decreases when it moves in the same direction as the wave.

The propagation of longitudinal waves in a rod with a moving boundary has been analyzed. In this case, accounting for deformations renders the problem nonlinear. It has been shown that the amplitude of deformation waves remains unchanged, despite the influence of boundary velocity and deformation magnitude.

The effect of damping forces on the oscillation amplitude during passage through resonance has been studied. Criteria in the form of inequalities have been obtained, defining regions where the boundary motion must be taken into account.

The applied value of the results lies in their potential use for solving a wide range of engineering problems [29–33], including: analysis of longitudinal and bending vibrations of shafts, beams, and rods with movable supports; reliability assessment of ropes in lifting systems and dynamic stability of strings, fibers, and tape transmissions; study of vibrations in tapes used in transport mechanisms, band saws, and flexible transmission elements; analysis of wire oscillations during the fabrication of rotational shells by winding; process control in cable production, rolling; reliability assessment of railway overhead contact systems, etc.

These types of problems are understudied and require further research. The presented results make it possible, already at the design stage, to prevent the occurrence of high-amplitude oscillations in mechanical systems with moving boundaries [34–37].

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Conflict of Interest Statement: **the authors declare no conflict of interest.**

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Конфликт интересов: ***авторы заявляют об отсутствии конфликта интересов.***

Все авторы прочитали и одобрили окончательный вариант рукописи.

Received / Поступила в редакцию 30.03.2025
Revised / Поступила после рецензирования 25.04.2025
Accepted / Принята к публикации 20.05.2025