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Multistage Grid-Characteristic Method of Increased Order of Accuracy for Acoustic Problems

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Abstract

Introduction. Seismic exploration is a widely used technology for locating hydrocarbon deposits. An important stage of this process is the simulation of seismic wave propagation in a geological model of the medium with specified physical characteristics. Due to the high computational cost of this problem, the acoustic approximation is widely used in practice, allowing for the correct description of longitudinal wave propagation. The most common approach to seismic modeling is the use of finite-difference schemes on staggered Cartesian computational grids. Despite their simplicity of implementation and high computational efficiency, such methods exhibit insufficient accuracy when modelling complex geological structures, including curvilinear interfaces between geological layers. A promising direction is the development of new high-order computational methods on curvilinear computational grids. This paper presents a stable fifth-order grid-characteristic method successfully applied to solving the problem of acoustic wave propagation in the two-dimensional case.

Materials and Methods. The study employs a grid-characteristic method with a fifth-degree interpolation polynomial constructed on an extended spatial stencil. A class of curvilinear grids is identified that makes it possible to retain the accuracy achieved when solving a one-dimensional problem. Furthermore, the use of a multistage splitting method allows the preservation of the scheme's order in both time and space for multidimensional formulations.

Results. The formulas of the computational algorithm are presented, the achievement of the declared convergence order is empirically confirmed, and wavefield patterns of the dynamic process are calculated.

Discussion. The results demonstrate lower numerical dissipation of the proposed computational algorithm. The trade-off for this improvement is a significant increase in computation time.

Conclusion. The developed computational algorithm ensures high accuracy in calculating seismic fronts, which is critically important for seismic exploration tasks in layered geological massifs.

Keywords: seismic exploration, seismic waves, mathematical modelling, curvilinear grids, acoustic medium, grid-characteristic method, operator splitting

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Оригинальное эмпирическое исследование

Многостадийный сеточно-характеристический метод повышенного порядка точности для задач акустики

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Аннотация

Введение. Сейсмическая разведка является широко применяемой технологией поиска месторождений углеводородов. Важным этапом данного процесса является расчёт распространения сейсмических волн в геологической модели среды с заданными физическими характеристиками. Ввиду высокой вычислительной сложности задачи на практике активно используется акустическое приближение, позволяющее корректно описать распространение продольных волн. Наиболее часто для сейсмического моделирования используются конечно-разностные схемы на сдвинутых кубических расчётных сетках. Несмотря на простоту их реализации и высокую вычислительную эффективность, такие подходы демонстрируют недостаточную точность при моделировании сложных геологических структур, включая криволинейные границы раздела геологических слоёв. Перспективным направлением является разработка новых вычислительных методов высокого порядка точности на криволинейных расчётных сетках. В настоящей работе представлен устойчивый сеточно-характеристический метод пятого порядка аппроксимации, успешно применённый для решения задачи о распространении акустических волн в двумерной постановке.

Материалы и методы. Используется сеточно-характеристический метод с интерполяционным полиномом пятой степени, построенном на расширенном пространственном шаблоне. Выделен класс криволинейных сеток, позволяющий сохранить достигнутую при решении одномерной задачи точность расчёта. При этом с помощью метода многошагового расщепления удается сохранить порядок схемы по времени и по пространству в многомерной постановке.

Результаты исследования. Представлены формулы вычислительного алгоритма, эмпирически подтверждено достижение заявленного порядка сходимости, рассчитаны волновые картины динамического процесса.

Обсуждение. Результаты расчётов демонстрируют меньшую численную диссипацию предложенного вычислительного алгоритма. Платой за это является значимое увеличение времени расчёта.

Заключение. Разработанный расчётный алгоритм обеспечивает высокую точность расчёта сейсмических фронтов, что критически важно в задачах сейсморазведки в слоистых геологических массивах.

Ключевые слова: сейсмическая разведка, сейсмические волны, математическое моделирование, криволинейные сетки, акустическая среда, сеточно-характеристический метод, операторное расщепление

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Introduction. Computer simulation of wavefields in heterogeneous media is widely used in geophysical research and plays a key role in solving problems of migration and inversion of seismic exploration data [1, 2]. Many different numerical methods applicable to solving the dynamic deformation problem of geological media have been developed by various research groups: the finite-difference method [3], the finite-element method [4], the discontinuous Galerkin method [5], and the spectral element method [6]. Among them, the finite-difference method remains the most frequently used in practice due to its ease of implementation and high computational efficiency.

One of the actively developing methods is the grid-characteristic method [7], which is used in the present work. In recent years, active development of modifications of the grid-characteristic method has been carried out on various types of computational grids: unstructured tetrahedral [8, 9], Cartesian [10–13], curvilinear structured [14], chimeric overlapping [15–17] for solving practical problems of seismic exploration [8, 10, 11, 15, 16], seismic resistance [12, 14], non-destructive testing of composite materials [18], and calculation of vibrations of railway tracks [13, 17].

In marine seismic exploration, seismic waves propagate both in an acoustic medium (water layer) and in an elastic medium (sea bottom and underlying geological massif). If the interface between the media is curvilinear, the finite-difference method encounters significant difficulties when attempting to correctly calculate the travel times of waves reflected from the bottom [19]. One possible way to improve the accuracy of specifying the geometry of the interface is to reduce the grid step, but this leads to a significant increase in the computational complexity of the problem [20]. To achieve a compromise between the increase in computational costs and a decrease in the accuracy of modelling, it is possible to combine the finite difference method with the coordinate transformation technique [21]. This approach is based on mapping a curvilinear computational grid coinciding with the layer boundary into a computationally convenient orthogonal grid using a sufficiently smooth coordinate transformation.

In this paper, we consider the problem of seismic wave propagation in an acoustic medium containing curvilinear layer interfaces. Using the inverse transformation from curvilinear to Cartesian coordinates allows us to apply a grid-characteristic method of increased accuracy order on an extended spatial stencil for a special class of computational grids. To eliminate the effect of reducing the order of approximation of a two-dimensional computational algorithm in time due to the use of coordinate-wise splitting, the method of multi-step operator splitting is used [22]. The computational experiments conducted confirm the high accuracy of calculations and the stability of the scheme when the standard Courant condition is met.

Materials and Methods. The dynamic behavior of a homogeneous acoustic medium under small deformations and in the absence of external volumetric forces is described by a hyperbolic system of equations of the form:

$$\begin{cases} \rho\dfrac{\partial\vec{v}}{\partial t}+\nabla p=\vec{0}, \\ \dfrac{\partial p}{\partial t}+\rho c^2\nabla\cdot\vec{v}=0. \end{cases}$$

The following notations are used here: ρ is the medium density; c is the P-wave propagation velocity; $p(x,y,z,t)$ is the pressure, $\vec{v}(x,y,z,t)=(u,v,w)^T$ is the velocity vector at the considered point of the acoustic medium. In the two-dimensional formulation of the problem considered in this paper, all the sought functions do not depend on the third spatial variable: $\dfrac{\partial\vec{v}}{\partial z}=\vec{0},\dfrac{\partial p}{\partial z}=0.$ Let the integration domain initially be covered by some curvilinear structural computational grid so that its sufficiently smooth mapping onto a uniform square computational region is possible (Fig. 1). Let the relationship between the original Cartesian coordinates x and y and transformed coordinates ξ and η is set explicitly as

$$\begin{cases} x=x(\xi,\eta), \\ y=y(\xi,\eta). \end{cases}$$

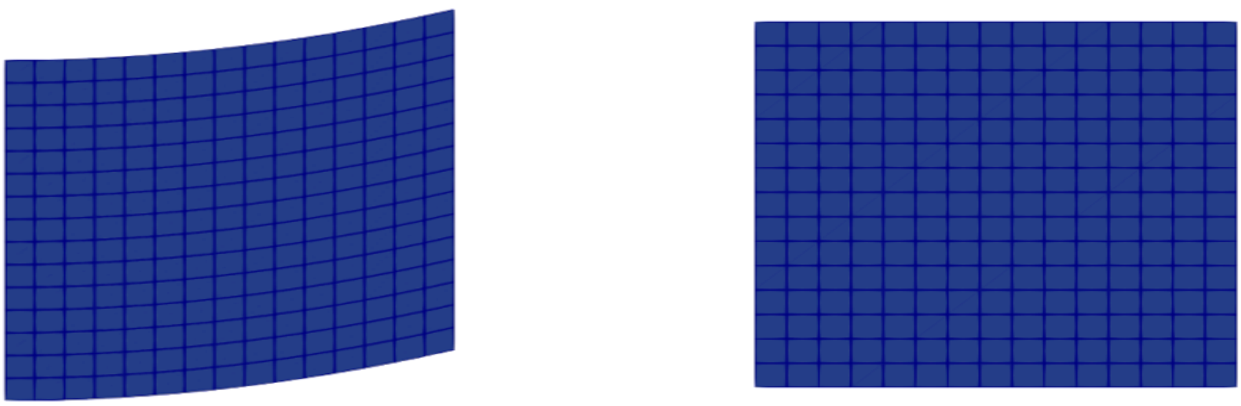

Fig. 1. The original curvilinear computational grid (x, y) (left) and the transformed square grid (ξ, η) (right)

By transitioning to new coordinates in the original system of equations, we obtain that

$$\frac{\partial p}{\partial t}+\rho c^2\frac{1}{|J|}\left[\left(\frac{\partial u}{\partial\xi}\frac{\partial y}{\partial\eta}-\frac{\partial u}{\partial\eta}\frac{\partial y}{\partial\xi}\right)\right]+\rho c^2\frac{1}{|J|}\left[\left(\frac{\partial v}{\partial\eta}\frac{\partial x}{\partial\xi}-\frac{\partial v}{\partial\xi}\frac{\partial x}{\partial\eta}\right)\right]=0,$$

$$\frac{\partial u}{\partial t}+\frac{1}{\rho}\frac{1}{|J|}\left(\frac{\partial p}{\partial \xi}\frac{\partial y}{\partial \eta}-\frac{\partial p}{\partial \eta}\frac{\partial y}{\partial \xi}\right)=0,$$

$$\frac{\partial v}{\partial t}+\frac{1}{\rho}\frac{1}{|J|}\left(\frac{\partial p}{\partial \eta}\frac{\partial x}{\partial \xi}-\frac{\partial p}{\partial \xi}\frac{\partial x}{\partial \eta}\right)=0,$$

where a new notation $|J|=\begin{vmatrix}\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi}\\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta}\end{vmatrix}$ is the Jacobi matrix.

This system of equations can be rewritten in canonical form:

$$\vec{q}_t+A_\xi\vec{q}_\xi+A_\eta\vec{q}_\eta=0.$$

The following additional notations are introduced here:

$$\vec{q}=(p,u,v)^T,$$

$$A_\xi=\begin{pmatrix}0 & \rho c^2\frac{1}{|J|}\frac{\partial y}{\partial \eta} & -\rho c^2\frac{1}{|J|}\frac{\partial x}{\partial \eta}\\ \frac{1}{\rho}\frac{1}{|J|}\frac{\partial y}{\partial \eta} & 0 & 0\\ -\frac{1}{\rho}\frac{1}{|J|}\frac{\partial x}{\partial \eta} & 0 & 0\end{pmatrix},$$

$$A_\eta=\begin{pmatrix}0 & -\rho c^2\frac{1}{|J|}\frac{\partial y}{\partial \xi} & \rho c^2\frac{1}{|J|}\frac{\partial x}{\partial \xi}\\ -\frac{1}{\rho}\frac{1}{|J|}\frac{\partial y}{\partial \xi} & 0 & 0\\ \frac{1}{\rho}\frac{1}{|J|}\frac{\partial x}{\partial \xi} & 0 & 0\end{pmatrix}.$$

To construct a numerical solution to this two-dimensional system of equations, one can use the method of splitting by spatial directions, thereby reducing the problem to a sequential solution of two one-dimensional problems

$$\vec{q}_t+A_\xi\vec{q}_\xi=0,$$

$$\vec{q}_t+A_\eta\vec{q}_\eta=0.$$

In this case, the solution of the first system of equations is the initial condition for solving the second system of equations. Note that this procedure allows us to construct a converging computational algorithm, which, however, in the general case has only the first order of approximation in time. This is due to the non-permutability of the operators associated with the matrices A_ξ and A_η.

Note that each of the one-dimensional systems with matrix $A_j (j=\xi,\eta)$ is hyperbolic and can be represented as follows:

$$A_j=\Omega_j^{-1}\Lambda_j\Omega_j,$$

$$\Lambda_\xi=\frac{1}{|J|}\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2+\left(\frac{\partial y}{\partial \eta}\right)^2}(c,-c,0),$$

$$\Lambda_\eta=\frac{1}{|J|}\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial y}{\partial \xi}\right)^2}(c,-c,0).$$

The solution of a one-dimensional hyperbolic system with constant coefficients can be reduced to the solution of a spatial interpolation problem by transitioning to Riemann invariants according to the formula:

$$\vec{\omega}=\Omega_j\vec{q}.$$

For clarity, let us consider the procedure for constructing a solution to the problem along the direction ξ. Performing left multiplication of a system of equations by a matrix Ω_ξ and substituting the expressions obtained above into the original system of equations, under the conditions of independence Ω_ξ from ξ, we obtain, that

$$\vec{\omega}_t + \Lambda_\xi \vec{\omega}_\xi = 0.$$

For each equation from this system of one-dimensional independent transport equations with constant coefficients, according to their characteristic properties, the value at the next time layer is exactly determined by the following expression:

$$\vec{\omega}\left(\xi_m, t^n + \tau\right) = \vec{\omega}\left(\xi_m - \Lambda_\xi \tau, t^n\right).$$

When calculating the right-hand side of this equality, the procedure of interpolation by polynomials of a given degree on a fixed spatial stencil is used. In this work, a grid-characteristic scheme of the fifth order of approximation is used, constructed on a seven-point template using an interpolation polynomial of the fifth order [23]. Then the desired vector function $\vec{q} = (p, u, v)^T$, on the next time layer can be calculated using the formula (due to the non-degeneracy of the transformation):

$$\vec{q} = \Omega_\xi^{-1} \vec{\omega}.$$

Note that the structure of the matrix of eigenvectors can be written compactly in tensor form. We introduce the following notations for the directions corresponding to the axes ξ, η:

$$\vec{n}_0 = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \eta}\right)^2 + \left(\frac{\partial y}{\partial \eta}\right)^2}} \begin{pmatrix} \frac{\partial y}{\partial \eta} \\ -\frac{\partial x}{\partial \eta} \end{pmatrix},$$

$$\vec{n}_1 = \frac{1}{\sqrt{\left(\frac{\partial x}{\partial \xi}\right)^2 + \left(\frac{\partial y}{\partial \xi}\right)^2}} \begin{pmatrix} -\frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \xi} \end{pmatrix}.$$

Then the transition to Riemann invariants has the form:

$$\omega_{1,2} = \frac{1}{2} \vec{n}_0 \cdot \vec{v} \pm \frac{p}{2\rho c}, for\, \xi,$$

$$\omega_{1,2} = \frac{1}{2} \vec{n}_1 \cdot \vec{v} \pm \frac{p}{2\rho c}, for\, \eta.$$

As noted earlier, the use of this splitting method reduces the order of approximation of the two-dimensional scheme in time. To solve this problem, this paper uses multi-step operator splitting based on the use of fractional time steps [24]. In general, this calculation algorithm for each time step can be represented as follows:

for *i* in 1 ,..., *s*:

solve step along ξ: step $\xi\left(\alpha_i^\xi \tau\right)$,

solve step along η: step $\eta\left(\alpha_i^\eta \tau\right)$.

Coefficients $\alpha_i^j, i \in (1,2,...,s), j \in (\xi,\eta)$, defining the values of fractional time steps, uniquely determine the multi-step splitting scheme. It should be noted that it is possible to construct a non-unique scheme of a given order of approximation with a given number of stages. In this paper, a 5th-order multi-step splitting scheme with 7 stages was used [24]. Its coefficients are presented in Table 1.

Table 1

Coefficients of the used 5th order multi-step splitting scheme

i	α_i^ξ	α_i^η
1	0.475018345144539497	–0.402020995028838599
2	0.021856594741098449	0.345821780864741783
3	–0.334948298035883491	0.400962967485371350

End of table 1

i	α_i^{ξ}	α_i^{η}
4	0.512638174652696736	0.980926531879316517
5	–0.011978701020553904	–1.362064898669775624
6	–0.032120004263046859	0.923805029000837468
7	0.369533888781149572	0.112569584468347105

Results. The constructed two-dimensional grid-characteristic scheme was applied to model the process of propagation of a plane *P*-wave. The problem statement typical for the field of seismic exploration in a subhorizontal layered geological massif was considered. The computational domain was covered by a curvilinear computational grid specified by the following coordinate transformation:

$$\begin{cases} x = \xi, \\ y = \eta + \gamma\xi^2, \end{cases}$$

where $\gamma = 5 \cdot 10^{-4}$.

The advantage of this parameterization method is the independence of the eigenvectors and eigenvalues of the problem for steps along (ξ, η) of (ξ, η) accordingly. This allows you to bring a matrix Ω_j under the sign of differentiation with respect to the coordinate and construct an exact solution to the one-dimensional problem. The initial area of interest in Cartesian coordinates occupied a square with a side of 600 m. The acoustic characteristics of the medium were set equal to the following values: the density of the medium $\rho = 1000$ kg/m^3, *P*-wave velocity $c = 2000$ m/s. The initial disturbance was set at a distance of 400 m from the lower boundary directed vertically downwards. Totally, 50ms of physical time were calculated. The time step was selected from the Courant stability condition for the intermediate step of the computational algorithm, corresponding to the maximum coefficient by modulus α_5^{η}.

To confirm the achievement of the declared increased order of convergence by this scheme, a series of calculations were carried out on successively refined curvilinear computational grids. The results of the empirical assessment of the order of convergence according to the norms L_1 and L_∞ are presented in Table 2.

Table 2

Study of the order of convergence of the constructed scheme. The problem with a vertical *P*-wave

h	Error in L_1	Error in L_∞	Order in L_1	Order in L_∞
2.000	2.6081E+09	9.0030E+05	–	–
1.000	1.0736E+09	3.6153E+05	1.281	1.316
0.500	1.1051E+08	4.3919E+04	3.280	3.041
0.250	4.3173E+06	1.6748E+02	4.678	4.713
0.125	1.3778E+05	5.3186E+01	4.970	4.977

The calculation was performed using the standard method of splitting by spatial directions. The results are presented in Table 3.

Table 3

Study of the order of convergence of the scheme with standard splitting. The problem with a vertical *P*-wave

h	Error in L_1	Error in L_∞	Order in L_1	Order in L_∞
2.000	1.7314E+09	5.9920E+05	–	–
1.000	4.3286E+08	1.6200E+05	2.000	1.887
0.500	3.5098E+07	1.9527E+04	3.624	3.052
0.250	5.6940E+06	3.8921E+03	2.624	2.327
0.125	1.3951E+06	9.6667E+02	2.029	2.009

Note that the propagation of the wavefront along one of the lines of the computational grid, as it happened in the test presented above, is not an essential requirement for maintaining the order of convergence by the scheme. Under the conditions described above, the problem of propagation of a *P*-wave at a fixed angle $\beta = -5°$ was solved. The results of the empirical assessment of the order of convergence for two norms are presented in Table 4.

Table 4

Study of the order of convergence of the constructed scheme. Problem with an inclined *P*-wave

h	Error in L_1	Error in L_∞	Order in L_1	Order in L_∞
2.000	2.1279E+09	8.9108E+05	–	–
1.000	8.7694E+08	3.5489E+05	1.279	1.328
0.500	8.8829E+07	4.2314E+04	3.303	3.068
0.250	3.4498E+06	1.6029E+03	4.686	4.722
0.125	1.1013E+05	5.0870E+01	4.969	4.978

The calculation was performed using the standard method of splitting by spatial directions. The results are presented in Table 5.

Table 5

Study of the order of convergence of the standard splitting scheme. The problem with an inclined *P*-wave

h	Error in L_1	Error in L_∞	Order in L_1	Order in L_∞
2.000	1.4481E+09	5.9613E+05	–	–
1.000	3.6522E+08	1.6221E+05	1.987	1.878
0.500	3.6355E+07	2.4925E+04	3.329	2.702
0.250	7.3132E+06	5.4214E+03	2.314	2.201
0.125	1.8012E+06	1.3417E+03	2.022	2.015

Of greatest interest is the calculation of the process of seismic wave propagation in a medium consisting of geological layers with different mechanical characteristics (sandstones, clays, carbonates). To test the possibility of using the developed numerical scheme to solve this type of problem, the following computational experiment was conducted. Three computational grids were considered, covering three geological layers occupying a physical area of 90×150 m. The computational grid in the middle area was set as curvilinear with the parameter $\gamma = 5 \cdot 10^{-4}$. This leads to the formation of a curvilinear upper and lower boundaries. Then, the computational grids in the upper and lower regions were set with a gradually changing parameter γ so that the upper boundary of the upper grid and the lower boundary of the lower grid remained horizontal. A test was performed for the absence of significant reflections from "virtual" geological boundaries caused only by dividing the entire computational grid into three subregions. For this purpose, the same acoustic parameters were used in each of the layers.

The seismic signal source was a plane *P*-wave propagating downwards at a distance of 20 m from the upper boundary of the upper subdomain. The spatial grid step was 0.5 m, the time step was 100μs, which satisfies the Courant stability condition. Spatial pressure distributions in the entire computational domain at a fixed time T = 50 ms, obtained using the widely used third-order approximation grid-characteristic scheme and standard splitting scheme, and using the fifth-order approximation grid-characteristic scheme and multi-step splitting scheme described in the work are presented in Fig. 2. The amplitude of the original wave is more accurately preserved and there are no significant reflections.

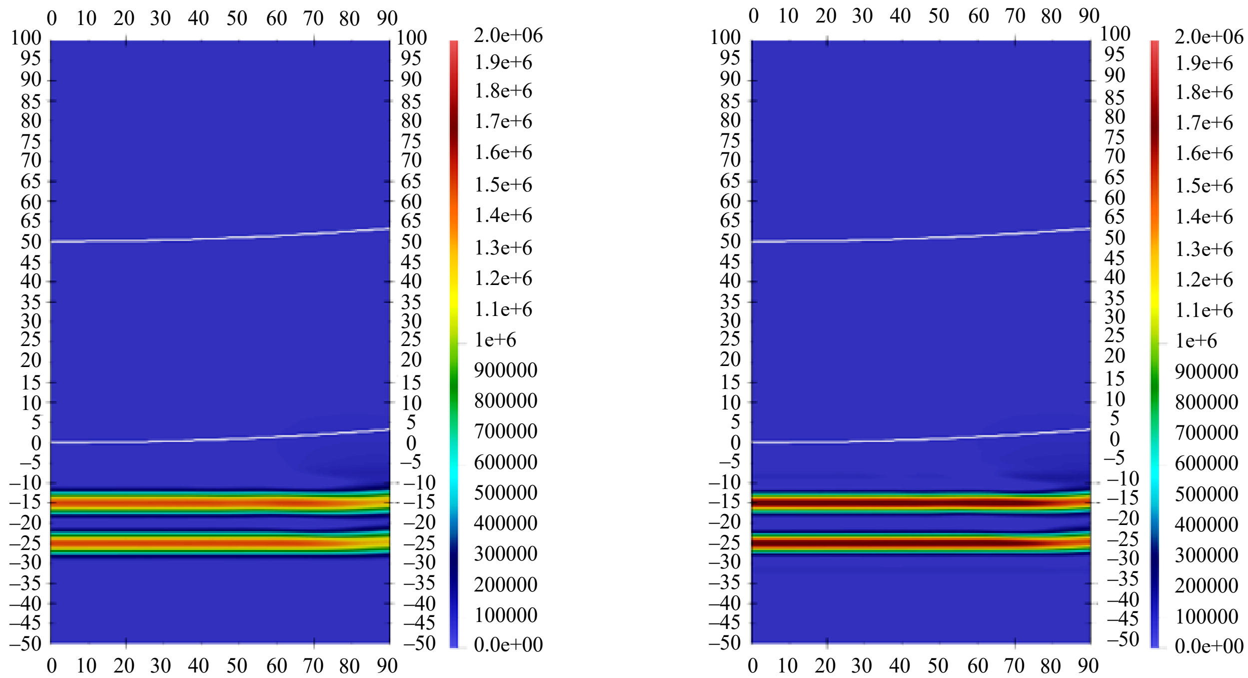

Fig. 2. Acoustic field in a three-layer medium. The following are used: a one-dimensional scheme of the third order of approximation and the usual spatial splitting (left) and the scheme proposed in this paper (right)

Discussion. In this paper, a new two-dimensional grid-characteristic scheme on curvilinear structural computational grids is presented. It is based on the application of the multistep splitting method to preserve the high order of approximation in time and uses the properties of the hyperbolic system of equations to reduce the solution of a one-dimensional hyperbolic problem to the procedure of spatial polynomial interpolation on a fixed seven-point template. The behavior of the obtained numerical solution to the problem of plane wave propagation in the computational domain covered by a special curvilinear grid is systematically investigated. The claimed 5th order of convergence in both coordinate and time is shown to be achieved. Note that in the case of impossibility of analytical calculation of the Jacobian of the transition between computational grids or the dependence of the operators of one-dimensional problems on the coordinate, the following modifications can be used. First, the Jacobian of the transition can be calculated with a given degree of accuracy by the finite-difference method. Secondly, the dependence of one-dimensional operators on the coordinate can be considered by using appropriate solvers of higher order of approximation for the one-dimensional hyperbolic problem.

The paper demonstrates the possibility of using the constructed simulation algorithm for modelling the seismic exploration process in a layered geological medium with curvilinear boundaries. To describe the horizontality of the daylight surface, a calculation grid is used that gradually levels out with distance from the interface. The comparison of the obtained acoustic wavefields with another calculation scheme showed the possibility of increasing the accuracy of preserving the amplitudes of propagating waves together with the absence of significant numerical artifacts at the contact boundaries.

Conclusion. Thus, it seems possible to apply the described approach to solving practical problems of seismic exploration. Promising areas of further research are:

1. Generalization of the simulation algorithm to more complex models of geological media: elastic, elastoplastic, elastoviscoplastic models;
2. Generalization of the calculation algorithm to three-dimensional problem statements to increase its universality and engineering applicability.

References

1. Kallivokas L.F., Fathi A., Kucukcoban S., Stokoe II K.H., Bielak J., Ghattas O. Site characterization using full waveform inversion. *Soil Dynamics and Earthquake Engineering*. 2013;47:62–82. http://dx.doi.org/10.1016/j.soildyn.2012.12.012

2. Yun Zhao, Xiaotao Wen, Chunlan Xie, Bo Li, Chenlong Li, Xiao Pan, et al. Simultaneous seismic inversion of effective stress parameter, fluid bulk modulus, and fracture density in TTI media. *Petroleum Science*. 2025. https://doi.org/10.1016/j.petsci.2025.04.002.

3. Samarskiy A.A., Nikolaev E.S. *Methods for solving grid equations*. Moscow: Nauka; 1978. 592 p. (In Russ.)

4. Matzen R. An efficient finite element time-domain formulation for the elastic second-order wave equation: A non-split complex frequency shifted convolutional PML. *International Journal for Numerical Methods in Engineering*. 2011;88(10):951–973. https://doi.org/10.1002/nme.3205

5. Antonietti P.F., Mazzieri I., Migliorini F. A discontinuous Galerkin time integration scheme for second order differential equations with applications to seismic wave propagation problems. *Computers & Mathematics with Applications*. 2023;134:87–100. https://doi.org/10.48550/arXiv.2112.01792

6. Komatitsch D., Tromp J. Introduction to the spectral element method for three-dimensional seismic wave propagation. *Geophysical Journal International*. 1999;139(3):806–822. https://doi.org/10.1046/J.1365-246X.1999.00967.X.

7. Magomedov K.M., Kholodov A.S. *Grid-characteristic numerical methods: textbook for universities*. 2nd ed., ispr. and add. Moscow: Yurait Publishing House; 2025. 313 p. (In Russ.)

8. Petrov I.B., Favorskaya A.V., Muratov M.V., Biryukov V.A., Sannikov A.V. Grid-characteristic method on unstructured tetrahedral grids. *Doklady Mathematics*. 2014;90:781–783. https://doi.org/10.1134/S1064562414070254

9. Favorskaya A.V., Petrov I.B. A study of high-order grid-characteristic methods on unstructured grids. *Numerical Analysis and Applications*. 2016;9:171–178. https://doi.org/10.1134/S1995423916020087

10. Favorskaya A.V., Zhdanov M.S., Khokhlov N.I., Petrov I.B. Modelling the wave phenomena in acoustic and elastic media with sharp variations of physical properties using the grid-characteristic method. *Geophysical Prospecting*. 2018;66(8):1485–1502. https://doi.org/10.1111/1365–2478.12639

11. Favorskaya A.V., Petrov I.B. The use of full-wave numerical simulation for the investigation of fractured zones. *Mathematical Models and Computer Simulations*. 2019;11:518–530. https://link.springer.com/article/10.1134/S2070048219040069

12. Favorskaya A.V., Petrov I.B. Inverse Problem of Determining the Strength Characteristics of Multi-Story Buildings on Piles. *Lobachevskii Journal of Mathematics*. 2025;46(1):202–213. https://doi.org/10.1134/S1995080224608555

13. Kozhemyachenko A.A., Petrov I.B., Favorskaya A.V., Khokhlov N.I. Boundary conditions for modeling the impact of wheels on railway track. *Computational Mathematics and Mathematical Physics*. 2020;60:1539–1554. https://link.springer.com/article/10.1134/S0965542520090110

14. Favorskaya A.V., Khokhlov N.I., Petrov I.B. Grid-characteristic method on joint structured regular and curved grids for modeling coupled elastic and acoustic wave phenomena in objects of complex shape. *Lobachevskii Journal of Mathematics*. 2020;41:512–525. https://doi.org/10.1134/S1995080220040083

15. Khokhlov N., Favorskaya A., Stetsyuk V, Mitskovets I. Grid-characteristic method using Chimera meshes for simulation of elastic waves scattering on geological fractured zones. *Journal of Computational Physics*. 2021;446:110637. https://doi.org/10.1016/j.jcp.2021.110637

16. Khokhlov N.I., Favorskaya A., Furgailo V. Grid-characteristic method on overlapping curvilinear meshes for modeling elastic waves scattering on geological fractures. *Minerals.* 2022;12(12):1597. https://doi.org/10.3390/min12121597

17. Pesnya E., Kozhemyachenko A.A., Favorskaya A.V. Vibration Analysis of Frost Heaving of the Ice Lens Type on Railways by a Grid-Characteristic Method. *Lobachevskii Journal of Mathematics*. 2025;46(1):317–325. https://doi.org/10.1134/S1995080224608270

18. Petrov I., Vasyukov A., Beklemysheva K., Ermakov A., Favorskaya A. Numerical modeling of non-destructive testing of composites. *Procedia Computer Science*. 2016;96:930–938. https://doi.org/10.1016/j.procs.2016.08.272

19. Yao Gang, da Silva N.V., Debens H.A., Wu Di. Accurate seabed modeling using finite difference methods. Computational Geosciences. 2018;22:469–84. https://doi.org/10.1007/s10596-017-9705-5

20. van Vossen R., Robertsson J.O., Chapman C.H. Finite-difference modeling of wave propagation in a fluid-solid configuration. *Geophysics*. 2002;67(2):618–24. https://doi.org/10.1190/1.1468623

21. Sun Yaochong, Zhang Wei, Xu Jiankuan, Chen Xiaofei. Numerical simulation of 2-D seismic wave propagation in the presence of a topographic fluid–solid interface at the sea bottom by the curvilinear grid finite-difference method. *Geophysical Journal International*. 2017;210(3):1721–1738. https://doi.org/10.1093/gji/ggx257

22. Golubev V.I., Shevchenko A.V., Petrov I.B. Raising convergence order of grid-characteristic schemes for 2D linear elasticity problems using operator splitting. *Computer Research and Modeling*. 2022;14(4):899–910. https://doi.org/10.20537/2076-7633-2022-14-4-899-910

23. Mi Xin, Golubev V. Two-Dimensional Grid-Characteristic Schemes for Acoustic Wave Simulations. Lobachevskii Journal of Mathematics. 2025;46:283–291. https://doi.org/10.1134/S199508022460852X

24. Auzinger W., Hofstätter H., Ketcheson D., Koch O. Practical splitting methods for the adaptive integration of nonlinear evolution equations. Part I: Construction of optimized schemes and pairs of schemes. *BIT. Numerical Mathematics.* 2017;57:55–74. https://doi.org/10.1007/s10543-016-0626-9

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