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Original Empirical Research

Application of Neural Networks to Steady-State Oscillations

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Abstract

Introduction. In recent years, the field of mathematics specializing in the application of artificial neural networks has been rapidly developing. In this work, a new method for constructing a neural network for solving wave differential equations is proposed. This method is particularly effective in solving boundary value problems for domains of complex geometric shapes.

Materials and Methods. A method is proposed for constructing a neural network designed to solve the wave equation in a planar domain G bounded by an arbitrary closed curve. It is assumed that the boundary conditions are periodic functions of time t, and the steady-state regime is considered. When constructing the neural network, the activation functions are taken as derivatives of singular solutions of the Helmholtz equation. The singular points of these solutions are uniformly distributed along closed curves surrounding the domain boundary. The training set consists of a set of particular solutions of the Helmholtz equation.

Results. Results were obtained for the solution of the first boundary value problem in various domains of complex geometric shape and under different boundary conditions. The results are presented in tables containing both the exact solutions of the problem and the solutions obtained using the neural network. A graphical comparison is also provided between the exact solution and the solution obtained with the constructed neural network.

Discussion. The presented computational results demonstrate the efficiency of the proposed method for constructing neural networks that solve boundary value problems of partial differential equations in domains of complex geometry.

Conclusion. The further development of the proposed method may be applied to solving boundary value problems for the wave equation in exterior domains. Of particular interest is the application of this method to diffraction problems.

Keywords: wave equation, domain of complex geometric shape, neural networks

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Оригинальное эмпирическое исследование

Применение нейронных сетей для решения задачи об установившихся колебаниях

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Аннотация

Введение. В последнее время быстро развивается область математики, специализирующаяся на применении искусственных нейронных сетей. В настоящей работе предложен новый метод построения нейронной сети для решения волновых дифференциальных уравнений. Этот метод особенно эффективен при решении краевых задач для областей сложной геометрической формы.

Материалы и методы. Предлагается метод построения нейронной сети, предназначенной для решения волнового уравнения для плоской области G, ограниченной произвольной замкнутой кривой. Предполагается, что граничные условия являются периодическими функциями времени *t*. Рассматривается установившийся режим. При построении нейронной сети в качестве активационных функций принимаются производные от сингулярных решений уравнения Гельмгольца. Сингулярные точки этих решений равномерно распределены по замкнутым кривым, охватывающим границу области. В качестве обучающего множества используется множество частных решений уравнения Гельмгольца.

Результаты исследования. Получены результаты решения первой краевой задачи для различных областей сложной геометрической формы и граничных условий. Результаты представлены в виде таблиц, содержащих точные решения задачи и решения, полученные с помощью нейронной сети. Дано графическое представление точного решения и решения, полученного построенной нейронной сетью.

Обсуждение. Представленные результаты расчетов показали эффективность предложенного метода построения нейронных сетей, решающих краевые задачи дифференциальных уравнений в частных производных для областей сложной геометрической формы.

Заключение. Дальнейшее развитие разработанного автором метода может быть применено к решению краевых задач для волнового уравнения, для решения внешних задач. Особенный интерес представляет применение этого метода к задачам дифракции.

Ключевые слова: волновое уравнение, область сложной геометрической формы, нейронные сети

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Introduction. In modelling various natural objects and phenomena, the apparatus of partial differential equations (PDEs) is often employed. The complexity of the developed models does not always allow for the efficient use of traditional methods. Therefore, neural network methods are increasingly being applied in recent years.

The theoretical foundations of the neural network method were laid in the mid-20th century in the works of A.N. Kolmogorov [1]. At present, neural networks are widely used in solving various boundary value problems. Works [2, 3] are devoted to solving boundary value problems for the Laplace equation. In [4], deep learning methods are applied to the Poisson equation in a two-dimensional domain. In [5], approaches to solving heat and mass transfer problems using perceptron-type neural networks are investigated.

Currently, physics-informed neural networks (PINNs) are frequently employed to solve partial differential equations [6, 7]. In [8, 9], radial basis functions are used as activation functions, with their parameters proposed to vary during the training process. Successful applications of neural networks to solving the Navier–Stokes equations are presented in [10, 11].

In [12], radial basis function neural networks are applied to solving direct and inverse scattering problems. The present work represents a further development of the method for constructing neural networks used to solve PDEs, as presented in [13–15]. The essence of this method lies in using functions that satisfy the considered differential equation as activation functions. In this paper, this approach is applied to the construction of a neural network designed to solve boundary value problems for the wave equation.

Materials and Methods. Let us consider the first boundary value problem for the wave equation in a planar domain G, bounded by an arbitrary smooth closed curve γ. Assume that the boundary conditions are periodic functions of time t with period ω, acting from the initial moment of time $t = -\infty$.

Then, the solution of the wave equation

$$\Delta U = c^{-2}\frac{\partial^2 U}{\partial t^2}$$

can be sought in the form:

$$U = U_1(x,y)\cos\omega t + U_2(x,y)\sin\omega t,$$

where the functions $U_1(x, y)$ and $U_2(x, y)$ satisfy the equation

$$\Delta U = \frac{\omega^2}{c^2}U.$$

To solve the boundary value problem for equation (1), the neural network constructed below was employed. In this case, the sought function U was represented as:

$$U(x,y) = \sum_{k=1}^{N} w_k u(s_k) F(x,y,x_o(\sigma_k),y_o(\sigma_k)) + \sum_{k=1}^{N} v_k u(s_k) G(x,y,x_o(\tau_k),y_o(\tau_k)),$$

where $u(s_k)$ are the prescribed values of the unknown function U on the boundary of the domain; $F(x, y, x_0(\sigma_k), y_0(\sigma_k))$ and $G(x, y, x_0(\tau_k), y_0(\tau_k))$ are activation functions; σ_k and τ_k are arc coordinates on the contours γ_1 and γ_2, obtained from the boundary contour γ by shifting each point in the direction of the outward normal to the boundary by the distances ρ_1 and ρ_2 respectively; x, y are the coordinates of the points in the domain G.

The activation functions were chosen as

$$F(x,y,x_o(\sigma_k),y_o(\sigma_k)) = \frac{\partial^4}{\partial x^2 \partial y^2} Y_0\left(\frac{\omega R}{c}\right)$$

and

$$G(x,y,x_o(\sigma_k),y_o(\sigma_k)) = \frac{\partial^5}{\partial x^3 \partial y^2} Y_0\left(\frac{\omega R}{c}\right) n_1(\sigma_k) + \frac{\partial^5}{\partial x^2 \partial y^3} Y_0\left(\frac{\omega R}{c}\right) n_2(\sigma_k),$$

where $R = \sqrt{(x-x_o)^2 + (y-y_o)^2}$; $n_1(\sigma_k)$, $n_2(\sigma_k)$ are the coordinates of the singular points uniformly distributed along the auxiliary contours γ_2; $Y_0(z)$ is the Bessel function of the second kind.

Since the activation functions satisfy equation (1), it remains only to fulfill the boundary conditions on the contour γ

$$U\big|_\gamma = u.$$

During the training of the network, the weights and the parameters ρ_1 and ρ_2, were determined by minimizing the error functional

$$\Phi(w_k, v_k, \rho_1, \rho_2) = \sum_{j=1}^{M} \sum_{i=1}^{N} \left\{ \sum_{k=1}^{N} w_k f_k^j F(x_i, y_i, x_o(\sigma_k), y_o(\sigma_k) + \right\}$$

$$v_k f_k^j G(x_i, y_i, x_o(\tau_k), y_o(\tau_k) - f_i^j \}^2,$$

where f_k^j is the value of the j-th function from the training set at the point of the boundary contour with coordinate σ_k.

To determine w_k and v_k from the obtained relations

$$\frac{\partial \Phi}{\partial w_k} = 0 \text{ and } \frac{\partial \Phi}{\partial v_k} = 0$$

a system of linear algebraic equations was solved. The values of ρ_1 and ρ_2 are determined by a simple search procedure.

The accuracy of the obtained solution can be evaluated by comparing the values of U on the boundary of the domain, computed using the neural network, with the prescribed boundary values:

$$U(x(s_i), y(s_i)) = \sum_{k=1}^{N} w_k u(s_k) F(s_i, \sigma_k) + \sum_{k=1}^{N} v_k u(s_k) G(s_i, \tau_k),$$

$$F(s_i, \sigma_k) = F(x(s_i), y(s_i), x_o(\sigma_k), y_o(\sigma_k)),$$

$$G(s_i, \tau_k) = F(x(s_i), y(s_i), x_o(\tau_k), y_o(\tau_k)).$$

As the training set, a collection of functions was used that are solutions of equation (1) and have the form:

$$v_1 = \cos\left(\frac{\omega}{c} n_1^k x + \alpha_k\right) \cos\left(\frac{\omega}{c} n_2^k y - \alpha_k\right),$$

$$v_2 = \cos\left(\frac{\omega}{c} n_2^k x + \alpha_k\right) \cos\left(\frac{\omega}{c} n_1^k y - \alpha_k\right).$$

where each function corresponds to a boundary point with index k. With a change in the index k the values of α_k, as well as the components of the normal vector were also varied.

The parameters determined by the method described above do not always provide the desired accuracy of the neural network solution. In such cases, the required accuracy can be achieved by iterative refinement of the result using the following algorithm:

$$\Delta u^o(s_i) = u(s_i), u_t^o(s_i) = u(s_i),\ i = 1,2,\ldots N,$$

$$\Delta V^{n+1}(s_i) = \sum_{k=1}^{N} \left\{ w_k \Delta u^n(s_k) F(s_i, \sigma_k) + v_k \Delta u^n(s_k) G(s_i, \tau_k) \right\},$$

$$\Delta u^{n+1}(s_i) = \Delta u^{n+1}(s_i) - \Delta V^{n+1}(s_i),$$

$$u_t^{n+1}(s_i) = u_t^{n+1}(s_i) - \Delta u^{n+1}(s_i).$$

where $u_t^{n+1}(s_i)$ is the refined solution at the boundary of the domain.

The iterative refinement continues until the specified accuracy $\frac{\left\|\Delta u^{n+1}(s_i)\right\|}{\left\|\Delta u_t^{n+1}(s_i)\right\|} < \delta$ is reached (where δ defines the desired precision of the function U on the boundary of the domain G), or until the value of $\frac{\left\|\Delta u^{n+1}(s_i)\right\|}{\left\|\Delta u_t^{n+1}(s_i)\right\|}$ begins to increase.

After that, the solution at any point in the domain G is calculated by the formula:

$$U(x,y) = \sum_{k=1}^{N} w_k u_t(s_k) F(x,y,x_o(\sigma_k),y_o(\sigma_k)) + \sum_{k=1}^{N} v_k u_t(s_k) G(x,y,x_o(\tau_k),y_o(\tau_k)).$$

Results. The method described above was applied to solving boundary value problems for the wave equation in planar domains whose boundaries were defined as

$$\begin{cases} x = a\cos(t) + g\sin(dt), \\ y = b\sin(t) + f\cos(dt). \end{cases}$$

where $t \in [0, 2\pi]$, a, b, g, f, d are adjustable parameters.

In all the problems considered below, the following parameter values were used: number of functions in the training set $M = 72$, number of neurons in the network $N = 72$, $c = 250$, $\delta = 0.0025$.

Problem 1. A planar domain was considered whose shape was determined by the parameters: $a = 0.27$, $b = 0.27$, $g = -0.055$, $f = 0.055$, $d = 3$ (Fig. 1).

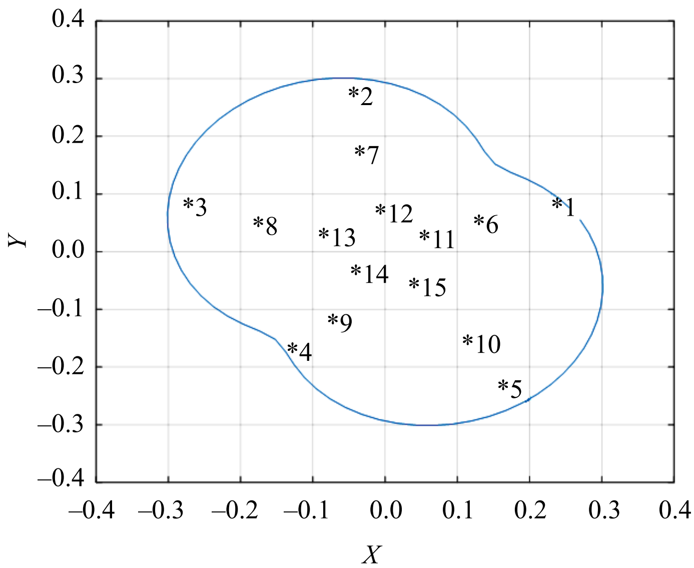

Fig. 1. Shape of the domain in Problem 1

In Fig. 1, the stars indicate the points of the domain where the exact solution values and the values obtained by the neural network were computed. Table 1 presents the computational results (amplitudes) corresponding to the solution, which in polar coordinates has the form:

$$U = J_1\left(\frac{\omega r}{c}\right)\cos\varphi\cos(\omega t),\ \omega = 550.$$

Table 1

Computational results for Problem 1

Point No.	1	2	3	4	5
Exact solution	0.25615	0.12741	–0.01673	–0.18688	–0.28990
Neural network	0.25598	0.12730	–0.01643	–0.18703	–0.28962
Point No.	6	7	8	9	10
Exact solution	0.16308	0.08066	–0.01072	–0.12037	0.18643
Neural network	0.16297	0.08061	–0.01065	–0.12032	0.18635
Point No.	11	12	13	14	15
Exact solution	0.06463	0.03187	–0.00426	–0.04800	–0.07427
Neural network	0.06457	0.03184	–0.00426	–0.04800	–0.07425

Problem 2. A planar domain was studied whose shape was determined by the pa-rameters: $a = 0.27$, $b = 0.27$, $g = –0.035$, $f = 0.035$, $d = 4$ (Fig. 2).

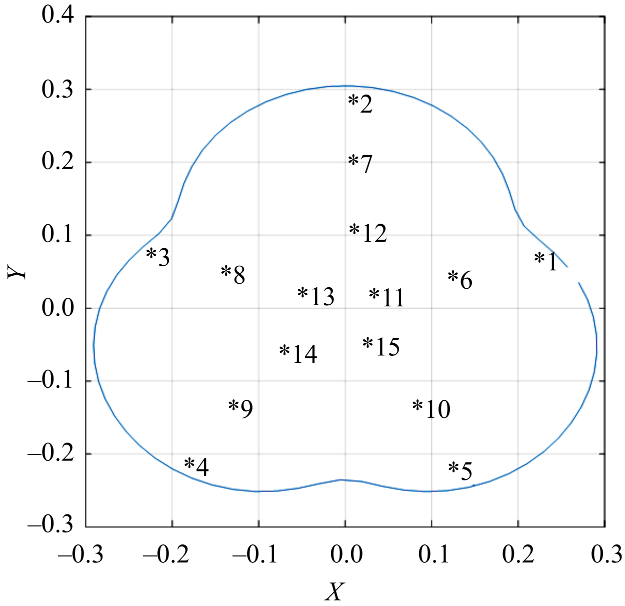

Fig. 2. Shape of the domain in Problem 2

The solution of the wave equation was considered in the form:

$$U = \cos\left(\left(\frac{\omega}{c}\right)\cdot\left(ct - \cos(f)x - \sin(f)y\right)\right),\ f = 1.5,\ \omega = 125,$$

where the given expression satisfies both the wave equation and equation (1). Therefore, time ttt was treated as a parameter: a fixed value was assigned to ttt, and then the algorithm for obtaining the solution described above was imple-mented. Table 2 presents the computational results obtained using the neural network, along with the exact solution of the problem corresponding to the time moment t=3T/5 (where T=2π/ω is the period of the solution).

Table 2

Computational results for Problem 2

Point No.	1	2	3	4	5
Exact solution	–0.82913	–0.86526	–0.88547	–0.87167	–0.83232
Neural network	–0.82908	–0.86525	–0.88553	–0.87169	–0.83227
Point No.	6	7	8	9	10
Exact solution	–0.82047	–0.84421	–0.86163	–0.84979	–0.82211
Neural network	–0.82046	–0.84420	–0.86164	–0.84979	–0.82210
Point No.	11	12	13	14	15
Exact solution	–0.81162	–0.82176	–0.83572	–0.82633	–0.81163
Neural network	–0.81161	–0.82176	–0.83572	–0.82633	–0.81163

Problem 3. A planar domain was studied whose shape was determined by the parameters: $a = 0.27$, $b = 0.27$, $g = 0.035$, $f = 0.035$, $d = 2$ (Fig. 3).

The solution of the wave equation was considered in polar coordinates in the form:

$$U = J_1\left(\frac{\omega r}{c}\right)\cos(\varphi - \omega t),\ \omega = 25.$$

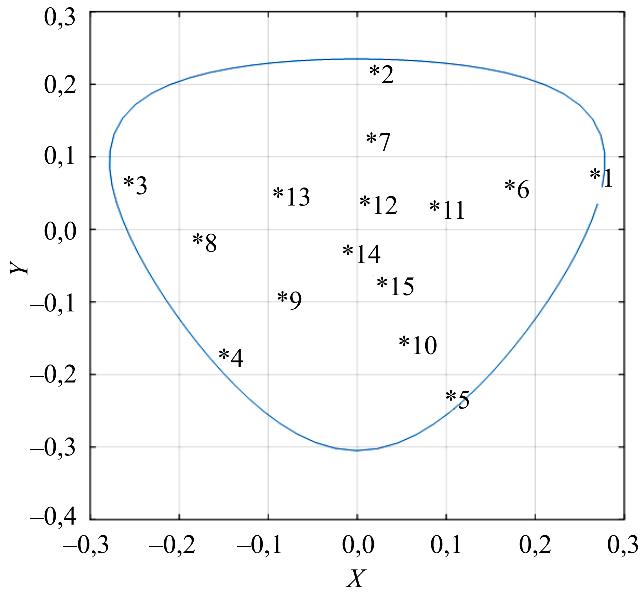

Fig. 3. Shape of the domain in Problem 3

Table 3 presents the computational results obtained using the neural network, together with the exact solution values of the problem corresponding to the time moment $t = 5T/10$.

Table 3

Computational results for Problem 3

Point No.	1	2	3	4	5
Exact solution	–0.01309	–0.01010	–0.002314	0.005639	0.01259
Neural network	–0.01309	–0.01010	–0.002317	0.005640	0.01259
Point No.	6	7	8	9	10
Exact solution	–0.00840	0.00672	–0.00145	0.00365	0.00840
Neural network	–0.00840	0.00672	–0.00146	0.00365	0.00840
Point No.	11	12	13	14	15
Exact solution	–0.00372	–0.00333	0.000595	0.00166	0.00421
Neural network	–0.00372	–0.00333	0.000594	0.00166	0.00421

Figures 4 and 5 show the time evolution of the solution at Points 3 and 5 obtained using the neural network. The stars indicate the exact solution values of the problem.

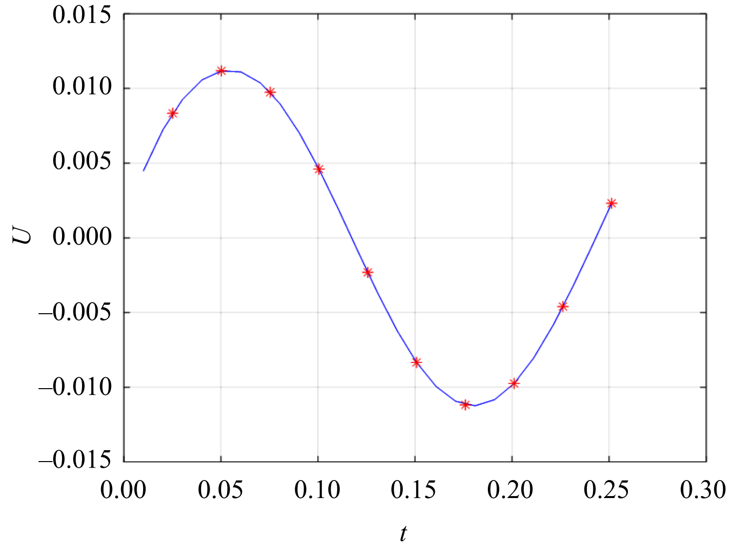

Fig. 4. Time evolution of the solution at Point 3

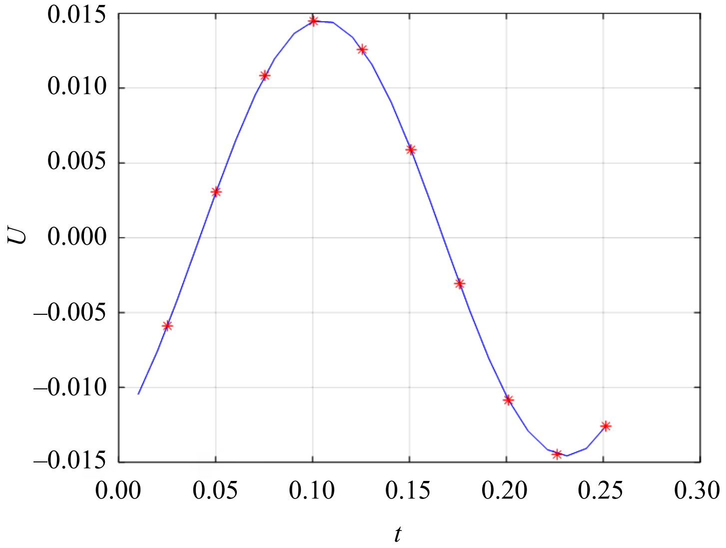

Fig. 5. Time evolution of the solution at Point 5

Discussion. The proposed method for constructing neural networks that solve boundary value problems of partial differential equations in domains of complex geometric shapes has demonstrated its effectiveness in the presented test problems.

Conclusion. Future research by the author will focus on applying the developed method to solving boundary value problems for the wave equation in exterior domains, as well as to diffraction problems. The development of this approach in the indicated directions may yield interesting and important results both in theory and in solving practical problems.

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