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Numerical Solution of Boundary Value Problems with a Homogeneous Linear Partial Differential Equation by a Modified Bubnov-Galerkin Method on a Rectangle

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Abstract

Introduction. This paper considers the numerical solution of a boundary value problem with a homogeneous linear partial differential equation of elliptic type on a rectangular domain using a modified Bubnov-Galerkin method. The unknown function is assumed to be zero at the vertices of the rectangle. Boundary value problems of this type include problems governed by the second-order Poisson and Laplace equations, the fourth-order biharmonic equation, and others. The developed algorithms allow equations with both constant and variable coefficients on the rectangular domain. The numerical solution of the boundary value problem is represented in a functional form, i. e., as a sum of coordinate functions with an unknown vector of expansion coefficients.

Materials and Methods. Due to the orthogonality of the proposed solution algorithms, the residual of the homogeneous linear equation is orthogonal over the rectangular domain to all coordinate functions of polynomial type, starting from the zero-index function that is identically equal to unity. All coordinate functions possess the unit Chebyshev norm on the rectangle. The inverse matrix required for solving the system of linear algebraic equations in the boundary value problem with the Poisson equation on the rectangle was computed using the Msimsl library.

Results. The numerical solution of the boundary value problem with the Poisson equation on a rectangle using the modified Bubnov-Galerkin method shows that the uniform Chebyshev norm of the residual of the boundary value problem is of order 10^{-12} and is comparable with the sweep method for a pentadiagonal matrix, in which the equation itself is approximated with eighth-order accuracy using the same number of nodes of a uniform grid. However, the computation time of the modified Bubnov-Galerkin method is 30 times smaller than that of the pentadiagonal sweep method for solving the same problem. High computational efficiency is the main advantage of the Bubnov-Galerkin methods for solving boundary value problems with partial differential equations without a noticeable loss of accuracy. The optimal number of coordinate functions in the problem is fourteen.

Discussion. The paper proposes algorithm (3)–(18) for solving the general problem with a homogeneous linear partial differential equation of arbitrary order *m*, and algorithm (19)–(26) for solving the Poisson (Laplace) equation on a rectangle using the modified Bubnov-Galerkin method. It should be noted that the symmetry of boundary conditions at the stage of reduction of the general problem leads to a decrease in the number of elementary subproblems. For the first time, an integral quadrature formula on a rectangle is proposed for computing the scalar product of two functions with twelfth-order accuracy.

Conclusion. Using algorithms (28) and (29), four examples of the Poisson equation on a rectangular domain were solved numerically, where the unknown function is zero at the vertices of the rectangle. An algorithm (5)–(19) was developed for solving boundary value problems with homogeneous elliptic partial differential equations of arbitrary order, demonstrating the successful application of the modified Bubnov-Galerkin method for solving boundary value problems on a rectangular domain.

Keywords: numerical methods, partial differential equations, hydrodynamics, initial-boundary value problem, mathematical modelling

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Оригинальное эмпирическое исследование

Численное решение краевых задач с однородным линейным уравнением в частных производных модифицированным методом Бубнова-Галеркина на прямоугольнике

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Аннотация

Введение. Рассматривается численное решение краевой задачи с линейным однородным уравнением в частных производных эллиптического типа на прямоугольнике модифицированным методом Бубнова-Галеркина, причем неизвестная функция равна нулю в вершинах прямоугольника. К краевым задачам данного типа можно отнести задачи с уравнениями Пуассона и Лапласа второго порядка, бигармоническое уравнение четвертого порядка и т. д. В полученных алгоритмах допускаются уравнения не только с постоянными, но и с переменными коэффициентами на прямоугольнике. Численное решение краевой задачи записывается в функциональном виде, то есть в виде суммы координатных функций с неизвестным вектором коэффициентов разложения.

Материалы и методы. В силу ортогональности предложенных алгоритмов решения задачи невязка линейного однородного уравнения ортогональна на прямоугольнике всем координатным функциям степенного вида, начиная с функции нуль индекса, тождественно равной единице. Все координатные функции имеют единичную норму Чебышева на прямоугольнике. Обратная матрица для решения системы линейных алгебраических уравнений в краевой задаче с уравнением Пуассона на прямоугольнике вычислялась библиотекой Msimsl.

Результаты исследования. Численное решение краевой задачи с уравнением Пуассона на прямоугольнике модифицированным методом Бубнова-Галеркина показало, что равномерная норма Чебышева невязки краевой задачи имеет порядок 10^{-12} и сравнима с методом прогонки с пятидиагональной матрицей, в котором само уравнение аппроксимировано с восьмым порядком погрешности и с тем же числом узлов равномерной сетки. Однако время решения краевой задачи модифицированным методом Бубнова-Галеркина в 30 раз меньше времени решения этой же задачи методом пятидиагональной прогонки. Быстродействие — главное преимущество методов Бубнова-Галеркина для решения краевых задач с уравнениями в частных производных без заметной потери точности решения. Оптимальным числом является четырнадцать координатных функций в задаче.

Обсуждение. В работе предложены алгоритм (3)–(18) для решения общей задачи с линейным однородным уравнением в частных производных произвольного порядка m и алгоритм (19)–(26) для решения уравнения Пуассона (Лапласа) на прямоугольнике модифицированным методом Бубнова-Галеркина. Отметим, что симметрия краевых условий на этапе редукции общей задачи приводит к уменьшению числа простых задач. Впервые предложена интегральная квадратурная формула на прямоугольнике для вычисления скалярного произведения двух функций с двенадцатым порядком погрешности.

Заключение. С помощью алгоритма (28), (29) численно решены четыре примера для уравнения Пуассона на прямоугольнике, причем неизвестная функция равна нулю в вершинах прямоугольника. Построен алгоритм (5)–(19) для решения краевых задач с однородными уравнениями в частных производных эллиптического типа произвольного порядка, что показывает успешное применение модифицированного метода Бубнова-Галеркина для решения краевых задач на прямоугольнике.

Ключевые слова: численные методы, уравнения в частных производных, гидродинамика, начально-краевая задача, математическое моделирование

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Introduction. This paper considers, for the first time, the application of a modified Bubnov-Galerkin method to solve boundary value problems on a rectangular domain. The method applies to homogeneous linear partial differential equations of elliptic type of arbitrary integer order m, with the unknown function assumed to be zero at the vertices of the rectangle. Methods for solving boundary value problems with linear partial differential equations and the classification of such equations are described in detail in [1–3]. Algorithms for solving boundary value problems, including two-dimensional problems, are presented in [4–10]. In [11–13], a modified Bubnov-Galerkin method was proposed for solving boundary value problems on an interval with an ordinary differential equation.

In the present work, the Bubnov-Galerkin method is generalized to solve boundary value problems on a rectangular domain with linear partial differential equations under the condition that the function vanishes at the vertices of the rectangle.

Materials and Methods

Problem statement. Consider a boundary value problem for a homogeneous linear partial differential equation of arbitrary order m on a rectangular domain for a function of two variables $u(x, y)$. An equation is called homogeneous if each of its terms is proportional to a partial derivative of the same order m. Elliptic-type equations of this class that do not depend on time include the Laplace equation ($m = 2$) [3, p. 29], the Poisson equation, and the biharmonic equation ($m = 4$) [3, p. 80]:

$$\begin{cases} Lu(x,y)=0,\ L\equiv\sum_{j=0}^{m} g_j(x,y)\dfrac{\partial^j\partial^{m-j}}{\partial x^j\partial y^{m-j}}, \quad x\in(a,b), y\in(c,d), \\ u\big|_{x=a}=\Phi_1(y), u\big|_{x=b}=\Phi_2(y), \\ u\big|_{y=c}=\Phi_3(x), u\big|_{y=d}=\Phi_4(x). \end{cases} \tag{1}$$

The coefficients of the linear equation (1) are assumed to be continuous $g_j(x,y)\in C[a,b]\times[c,d]$ and independent of the unknown function $u(x, y)$. The boundary conditions of problem (1) are continuous functions that satisfy homogeneous compatibility conditions at the vertices of the rectangle:

$$\Phi_1(c)=\Phi_3(a)=\Phi_1(d)=\Phi_4(a)=\Phi_2(c)=\Phi_4(b)=\Phi_2(d)=\Phi_3(b)=0. \tag{2}$$

Following [11–13], we choose a system of linearly independent functions on the interval $[a, b]$ or on the interval $[c, d]$:

$$\phi_k(x)=\left(\frac{2x-a-b}{b-a}\right)^k (k=\overline{0,n});\ \phi_k: x\in[a,b]\to z\in\begin{cases}[-1,1],\ k=2l+1,\\ [0,1],\ k=2l,\end{cases} \tag{3}$$

$$\phi_k(y)=\left(\frac{2y-c-d}{d-c}\right)^k (k=\overline{0,n});\ \phi_k: y\in[c,d]\to z\in\begin{cases}[-1,1],\ k=2l+1,\\ [0,1],\ k=2l.\end{cases} \tag{4}$$

Due to the linearity of equation (1), a reduction of the boundary value problem (1) can be performed; that is, its solution can be represented in the form $u(x,y)=u^1(x,y)+u^2(x,y)+u^3(x,y)+u^4(x,y)$, where:

$$\begin{cases} Lu^1(x,y)=0, x\in(a,b), y\in(c,d), \\ u^1\big|_{x=a}=\Phi_1(y), u^1\big|_{x=b}=0, \Phi_1(c)=\Phi_1(d)=0, \\ u^1\big|_{y=c}=0, u^1\big|_{y=d}=0, \end{cases} \tag{5.1}$$

$$\begin{cases} Lu^2(x,y)=0, x\in(a,b), y\in(c,d), \\ u^2\big|_{x=a}=0, u^2\big|_{x=b}=\Phi_2(y), \Phi_2(c)=\Phi_2(d)=0, \\ u^2\big|_{y=c}=0, u^2\big|_{y=d}=0, \end{cases} \tag{5.2}$$

$$\begin{cases} Lu^3(x,y)=0, x\in(a,b), y\in(c,d), \\ u^3\big|_{x=a}=0, u^3\big|_{x=b}=0, \Phi_3(a)=\Phi_3(b)=0, \\ u^3\big|_{y=c}=\Phi_3(x), u^3\big|_{y=d}=0, \end{cases} \tag{5.3}$$

$$\begin{cases} Lu^4(x,y)=0, x\in(a,b), y\in(c,d), \\ u^4\big|_{x=a}=0, u^4\big|_{x=b}=0, \Phi_4(a)=\Phi_4(b)=0, \\ u^4\big|_{y=c}=0, u^4\big|_{y=d}=\Phi_4(x). \end{cases} \tag{5.4}$$

The solution of problem (5.1) is sought in the form

$$u^1(x,y)=\Phi_1(y)X_1(x)=\Phi_1(y)\sum_{k=0}^{n}C_k^1\phi_k(x) \tag{6}$$

with an unknown vector of expansion weighting coefficients C_k^1.

Taking into account the explicit form of the partial differential operator in problem (1), we substitute the expansion of the solution (6) into the system of equations (5.1):

$$\sum_{j=0}^{m}g_j(x,y)\frac{\partial^j\partial^{m-j}}{\partial x^j\partial y^{m-j}}\left[\Phi_1(y)\sum_{k=0}^{n}C_k^1\phi_k(x)\right]=0 \Leftrightarrow \sum_{k=0}^{n}C_k^1\sum_{j=0}^{k}g_j(x,y)\Phi_1^{(m-j)}(y)\phi_k^{(j)}(x)=0.$$

In the last sum, the set of indices $0\le j\le k$ is explained by the fact that for coordinate functions of polynomial type the derivatives are given by

$$\phi_k^{(j)}(x)=\begin{cases}\left(\dfrac{2}{b-a}\right)^j k\cdot(k-1)\cdot\ldots\cdot(k-j+1)\left(\dfrac{2x-a-b}{b-a}\right)^{k-j}, \ j\le k. \\ 0, \ j>k\end{cases} \tag{7.1}$$

Similarly,

$$\phi_k^{(j)}(y)=\begin{cases}\left(\dfrac{2}{d-c}\right)^j k\cdot(k-1)\cdot\ldots\cdot(k-j+1)\left(\dfrac{2y-c-d}{d-c}\right)^{k-j}, \ j\le k. \\ 0, \ j>k\end{cases} \tag{7.2}$$

Let us multiply the last equation by the coordinate function $\phi_i(x)$ and, according to the Bubnov-Galerkin method, require the orthogonality of the residual of the equation to all coordinate functions $\phi_i(x),\ i=\overline{0,n-2}$. In this case, the residual coincides with the left-hand side of the last equation. After taking the scalar product of the functions in the metric of the functional space L_2 we obtain:

$$\sum_{k=0}^{n}C_k^1\left\langle\sum_{j=0}^{k}g_j(x,y)\Phi_1^{(m-j)}(y)\phi_k^{(j)}(x),\phi_i(x)\right\rangle=0. \tag{8}$$

The solution of problem (5.1) should be considered on the set of admissible functions using the boundary conditions (5.1), i. e.

$$u^1(a,y)=\Phi_1(y)\sum_{k=0}^{n}C_k^1\phi_k(a)=\Phi_1(y)\sum_{k=0}^{n}C_k^1(-1)^k=\Phi_1(y)\Leftrightarrow\sum_{k=0}^{n}C_k^1(-1)^k=1,$$

$$u^1(b,y)=\Phi_1(y)\sum_{k=0}^{n}C_k^1\phi_k(b)=\Phi_1(y)\sum_{k=0}^{n}C_k^1(1)^k=0\Leftrightarrow\sum_{k=0}^{n}C_k^1=0,$$

$$u^1(x,c)=\Phi_1(c)\sum_{k=0}^{n}C_k^1\phi_k(x)=0\Leftrightarrow\Phi_1(c)=0, u^1(x,d)=\Phi_1(d)\sum_{k=0}^{n}C_k^1\phi_k(x)=0\Leftrightarrow\Phi_1(d)=0.$$

As a result, we obtain a system of $n+1$ linear algebraic equations (SLAE) for the unknown coefficients C_k^1:

$$\begin{cases}\sum_{k=0}^{n}A_{i,k}^1C_k^1=f_i^1,\ i=\overline{0,n}, \\ A_{i,k}^1=\left\langle\sum_{j=0}^{k}g_j(x,y)\Phi_2^{(m-j)}(y)\phi_k^{(j)}(x),\phi_i(x)\right\rangle, i=\overline{0,n-2},\ k=\overline{0,n}, \\ A_{n-1,k}^1=(-1)^k, k=\overline{0,n};\quad A_{n,k}^1=1, k=\overline{0,n}, \\ f_i^1=0, i=\overline{0,n-2},\ f_{n-1}^1=1, f_n^1=0.\end{cases} \tag{9}$$

The nonhomogeneity of the SLAE is ensured by the coefficient in the right-hand side of equation (9).

In classical methods of mathematical physics, the solution of a boundary value problem is expanded in terms of eigenfunctions of the corresponding homogeneous boundary value problem, where each eigenfunction satisfies homogeneous boundary conditions. The homogeneity conditions for each eigenfunction $\phi_k(x)$ are sufficient for the entire sum of the form (6) to vanish at the endpoints of the interval $x = a$, $x = b$ but they are not necessary conditions. The necessary conditions in the present problem are $\sum_{k=0}^{n} C_k^1 (-1)^k = 1$ and $\sum_{k=0}^{n} C_k^1 = 0$.

Similarly, for problem (5.2) we obtain a solution of the form:

$$u^2(x,y) = \Phi_2(y)X_2(x) = \Phi_2(y)\sum_{k=0}^{n} C_k^2 \phi_k(x), \tag{10}$$

with an unknown vector of expansion weighting coefficients C_k^2 and the following system of linear algebraic equations:

$$u^2(a,y) = \Phi_2(y)\sum_{k=0}^{n} C_k^2 \phi_k(a) = \Phi_2(y)\sum_{k=0}^{n} C_k^2 (-1)^k = 0 \Leftrightarrow \sum_{k=0}^{n} C_k^2 (-1)^k = 0,$$

$$u^2(b,y) = \Phi_2(y)\sum_{k=0}^{n} C_k^2 \phi_k(b) = \Phi_2(y)\sum_{k=0}^{n} C_k^2 (1)^k = \Phi_2(y) \Leftrightarrow \sum_{k=0}^{n} C_k^2 = 1,$$

$$u^2(x,c) = \Phi_2(c)\sum_{k=0}^{n} C_k^2 \phi_k(x) = 0 \Leftrightarrow \Phi_2(c) = 0, u^2(x,d) = \Phi_2(d)\sum_{k=0}^{n} C_k^2 \phi_k(x) = 0 \Leftrightarrow \Phi_2(d) = 0,$$

$$\begin{cases} \sum_{k=0}^{n} A_{i,k}^2 C_k^2 = f_i^2,\ i = \overline{0,n}, \\ A_{i,k}^2 = \left\langle \sum_{j=0}^{k} g_j(x,y)\Phi_1^{(m-j)}(y)\phi_k^{(j)}(x), \phi_i(x) \right\rangle, i = \overline{0,n-2},\ k = \overline{0,n}, \\ A_{n-1,k}^2 = (-1)^k, k = \overline{0,n};\quad A_{n,k}^2 = 1, k = \overline{0,n}, \\ f_i^2 = 0, i = \overline{0,n-2},\ f_{n-1}^2 = 0, f_n^2 = 1. \end{cases} \tag{11}$$

For solving problem (5.3), the solution expansion with respect to the unknown vector of coefficients is written as C_k^3:

$$u^3(x,y) = \Phi_3(x)Y_3(y) = \Phi_3(x)\sum_{k=0}^{n} C_k^3 \phi_k(y) \tag{12}$$

with the boundary conditions

$$u^3(x,c) = \Phi_3(x)\sum_{k=0}^{n} C_k^3 \phi_k(c) = \Phi_3(x)\sum_{k=0}^{n} C_k^3 (-1)^k = \Phi_3(x) \Leftrightarrow \sum_{k=0}^{n} C_k^3 (-1)^k = 1,$$

$$u^3(x,d) = \Phi_3(x)\sum_{k=0}^{n} C_k^3 \phi_k(d) = \Phi_3(x)\sum_{k=0}^{n} C_k^3 (1)^k = 0 \Leftrightarrow \sum_{k=0}^{n} C_k^3 = 0,$$

$$u^3(a,y) = \Phi_3(a)\sum_{k=0}^{n} C_k^2 \phi_k(y) = 0 \Leftrightarrow \Phi_3(a) = 0, u^3(b,y) = \Phi_3(b)\sum_{k=0}^{n} C_k^3 \phi_k(y) = 0 \Leftrightarrow \Phi_3(b) = 0$$

and the following system of linear algebraic equations

$$\begin{cases} \sum_{k=0}^{n} A_{i,k}^3 C_k^3 = f_i^3,\ i = \overline{0,n}, \\ A_{i,k}^3 = \left\langle \sum_{j=m-k}^{m} g_j(x,y)\Phi_3^{(j)}(x)\phi_k^{(m-j)}(y), \phi_i(y) \right\rangle, i = \overline{0,n-2},\ k = \overline{0,n}, \\ A_{n-1,k}^3 = (-1)^k, k = \overline{0,n};\quad A_{n,k}^3 = 1, k = \overline{0,n}, \\ f_i^3 = 0, i = \overline{0,n-2},\ f_{n-1}^3 = 1, f_n^3 = 0. \end{cases} \tag{13}$$

Similarly, for problem (5.4), the solution is expanded in terms of the basis functions with a vector of weighting coefficients C_k^4:

$$u^4(x,y) = \Phi_4(x)Y_4(y) = \Phi_4(x)\sum_{k=0}^{n} C_k^4 \phi_k(y) \tag{14}$$

with the boundary conditions (the set of admissible functions)

$$u^4(x,c) = \Phi_4(x)\sum_{k=0}^{n} C_k^4 \phi_k(c) = \Phi_4(x)\sum_{k=0}^{n} C_k^4 (-1)^k = 0 \Leftrightarrow \sum_{k=0}^{n} C_k^4 (-1)^k = 0,$$

$$u^4(x,d) = \Phi_4(x)\sum_{k=0}^{n} C_k^4 \phi_k(d) = \Phi_4(x)\sum_{k=0}^{n} C_k^4 (1)^k = \Phi_4(x) \Leftrightarrow \sum_{k=0}^{n} C_k^4 = 1,$$

$$u^4(a,y)=\Phi_4(a)\sum_{k=0}^{n}C_k^4\phi_k(y)=0\Leftrightarrow\Phi_4(a)=0, u^4(b,y)=\Phi_4(b)\sum_{k=0}^{n}C_k^4\phi_k(y)=0\Leftrightarrow\Phi_4(b)=0$$

and the system of linear algebraic equations

$$\begin{cases}\sum_{k=0}^{n}A_{i,k}^4C_k^4=f_i^4,\ i=\overline{0,n},\\ A_{i,k}^4=\left\langle\sum_{j=m-k}^{m}g_j(x,y)\Phi_4^{(j)}(x)\phi_k^{(m-j)}(y),\phi_i(y)\right\rangle, i=\overline{0,n-2},\ k=\overline{0,n},\\ A_{n-1,k}^4=(-1)^k, k=\overline{0,n};\quad A_{n,k}^4=1, k=\overline{0,n},\\ f_i^4=0, i=\overline{0,n-2},\ f_{n-1}^4=0, f_n^4=1.\end{cases}\tag{15}$$

The general solution of problem (1) can be represented in the form

$$\begin{aligned}u(x,y)&=u^1(x,y)+u^2(x,y)+u^3(x,y)+u^4(x,y)=\\&=\Phi_1(y)\sum_{k=0}^{n}C_k^1\phi_k(x)+\Phi_2(y)\sum_{k=0}^{n}C_k^2\phi_k(x)+\Phi_3(x)\sum_{k=0}^{n}C_k^3\phi_k(y)+\Phi_4(x)\sum_{k=0}^{n}C_k^4\phi_k(y)=\\&=\sum_{k=0}^{n}\left(C_k^1\Phi_1(y)+C_k^2\Phi_2(y)\right)\phi_k(x)+\sum_{k=0}^{n}\left(C_k^3\Phi_3(x)+C_k^4\Phi_4(x)\right)\phi_k(y).\end{aligned}\tag{16}$$

Remark. If the boundary functions on the opposite sides of the rectangle are equal $\Phi_1(y)=\Phi_2(y)$, $\Phi_3(x)=\Phi_4(x)$, then, denoting the sum of the expansion coefficients by $C_k^1+C_k^2=C_k^*$, $k=\overline{0,n}$, $C_k^3+C_k^4=C_k^{**}$, $k=\overline{0,n}$ and adding the equations containing $C_k^1, C_k^2, C_k^3, C_k^4$, in systems (9), (11) and (13), (15), we obtain two systems of linear algebraic equations:

$$\begin{cases}\sum_{k=0}^{n}A_{i,k}^*C_k^*=f_i^*,\ i=\overline{0,n},\\ A_{i,k}^*=\left\langle\sum_{j=0}^{k}g_j(x,y)\Phi_1^{(m-j)}(y)\phi_k^{(j)}(x),\phi_i(x)\right\rangle, i=\overline{0,n-2},\ k=\overline{0,n},\\ A_{n-1,k}^*=(-1)^k, k=\overline{0,n};\quad A_{n,k}^*=1, k=\overline{0,n},\\ f_i^*=0, i=\overline{0,n-2},\ f_{n-1}^*=1, f_n^*=1,\end{cases}\tag{17.1}$$

$$\begin{cases}\sum_{k=0}^{n}A_{i,k}^{**}C_k^{**}=f_i^{**}{}_i,\ i=\overline{0,n},\\ A_{i,k}^{**}=\left\langle\sum_{j=m-k}^{m}g_j(x,y)\Phi_3^{(j)}(x)\phi_k^{(m-j)}(y),\phi_i(y)\right\rangle, i=\overline{0,n-2},\ k=\overline{0,n},\\ A_{n-1,k}^{**}=(-1)^k, k=\overline{0,n};\quad A_{n,k}^{**}=1, k=\overline{0,n},\\ f_i^{**}=0, i=\overline{0,n-2},\ f_{n-1}^{**}=1, f_n^{**}=1.\end{cases}\tag{17.2}$$

The solution given by the general formula (16) then transforms into the simplified form:

$$u(x,y)=\sum_{k=0}^{n}\left(C_k^*\Phi_1(y)\phi_k(x)+C_k^{**}\Phi_3(x)\phi_k(y)\right).\tag{18}$$

It should be noted that in the systems of linear algebraic equations (9), (11), (13), and (15), the scalar product of two functions over the rectangle is evaluated using a weight matrix corresponding to a two-dimensional quadrature integration formula with twelfth-order accuracy. The one-dimensional version of the weight coefficients was used in [11–13]:

$$\begin{aligned}\langle f_1,f_2\rangle&=\int_a^b\int_c^d f_1(x,y)f_2(x,y)dxdy=25h_1h_2\sum_{i=0}^{n_2}\sum_{j=0}^{n_1}C_{i,j}f_1\left(x_j,y_i\right)f_2\left(x_j,y_i\right)+\\&+O\left(h_1^{12}+h_2^{12}\right), C_{i,j}=C_i^2\cdot C_j^1, i=\overline{0,n_2}, j=\overline{0,n_1},\end{aligned}$$

$$x_j=a+jh_1, y_i=c+ih_2, n_1=10p_1, n_2=10p_2, h_1=\frac{b-a}{n_1}, h_2=\frac{d-c}{n_2}, p_1,p_2\in N,$$

where

$$C_i^k = \begin{cases} \dfrac{16067}{299376}, \ if\ i=0\ \ or\ i=n_k, k=1,2, \left(i=\overline{0,n_k}\right), \\ \dfrac{16067}{149688}, \ if\ \left(i\equiv 0 \bmod 10\right) and \left(0<i<n_k\right), \\ \dfrac{26575}{74844}, \ if\ \left(i\equiv 1 \bmod 10\right) or \left(i\equiv 9 \bmod 10\right), \\ \dfrac{-16175}{99792}, \ if\ \left(i\equiv 2 \bmod 10\right) or \left(i\equiv 8 \bmod 10\right), \\ \dfrac{5675}{6237}, \ if\ \left(i\equiv 3 \bmod 10\right) or \left(i\equiv 7 \bmod 10\right), \\ \dfrac{-4825}{5544}, \ if\ \left(i\equiv 4 \bmod 10\right) or \left(i\equiv 6 \bmod 10\right), \\ \dfrac{17807}{12474}, \ if\ i\equiv 5 \bmod 10. \end{cases} \quad (19)$$

Let us consider the numerical solution of the Poisson equation on a rectangle (the example is taken from [14]) using the algorithm obtained for solving the general problem (1):

$$\begin{cases} u_{xx}+u_{yy}=\sin x,\ 0<x<\pi, 0<y<\pi, \\ u\big|_{x=0}=u\big|_{x=\pi}=\sin y, 0\le x\le \pi, \\ u\big|_{y=0}=u\big|_{y=\pi}=\sin x, 0\le y\le \pi. \end{cases} \quad (20)$$

The example (20) has an exact analytical solution [14, p. 64]:

$$u(x,y)=\left(\left(\frac{1-\operatorname{ch}\pi}{\operatorname{sh}\pi}\right)\operatorname{sh}x+\operatorname{ch}x\right)\sin y+\left(\left(\frac{1-\operatorname{ch}\pi}{\operatorname{sh}\pi}\right)\operatorname{sh}y+\operatorname{ch}y\right)\sin x+\left(-1+\left(\frac{1-\operatorname{ch}\pi}{\operatorname{sh}\pi}\right)\operatorname{sh}y+\operatorname{ch}y\right)\sin x. \quad (21)$$

For the Poisson equation, the coefficients of the partial differential equation in problem (1) are equal $g_0(x,y)\equiv 1, g_1(x,y)\equiv 0, g_2(x,y)\equiv 1.$ The numerical solution of Example 1, taking into account taken the remark regarding the equality of the boundary conditions, as well as the equality of the right-hand side of the Poisson equation to the second boundary condition (the function $\sin x$) will be divided into two parts. The original problem (20) is reduced to two simpler problems, where the first problem has the form:

$$\begin{cases} u^1{}_{xx}+u^1{}_{yy}=\sin x,\ 0<x<\pi, 0<y<\pi, \\ u^1\big|_{x=0}=u^1\big|_{x=\pi}=0,\ 0\le x\le \pi, \\ u^1\big|_{y=0}=u^1\big|_{y=\pi}=\sin x,\ 0\le y\le \pi, \end{cases} \quad (22)$$

$$u^1(x,y)=\sin x\cdot\sum_{k=0}^{n}C_k^1\phi_k(y)$$

with the boundary conditions

$$y=0:\sum_{k=0}^{n}C_k^1\left(-1\right)^k=1, \quad y=\pi:\sum_{k=0}^{n}C_k^1=1.$$

Substituting the solution $u^1(x, y)$ into the Laplace equation, we obtain

$$-\sin x\cdot\sum_{k=0}^{n}C_k^1\phi_k(y)+\sin x\cdot\sum_{k=2}^{n}C_k^1\phi_k''(y)=\sin x \Leftrightarrow -C_0^1-C_1^1\phi_1+\sum_{k=2}^{n}C_k^1\left(\phi_k''-\phi_k\right)=1,$$

taking the scalar product of the last equation with the basis function $\phi_i(y), i=\overline{0,n-2}$, we obtain:

$$-C_0^1\left\langle 1,\phi_i\right\rangle-C_1^1\left\langle \phi_1,\phi_i\right\rangle+\sum_{k=2}^{n}C_k^1\left\langle \phi_k''-\phi_k,\phi_i\right\rangle=\left\langle 1,\phi_i\right\rangle, i=\overline{0,n-2} \Leftrightarrow \sum_{k=0}^{n}A_{i,k}^1C_k^1=f_i^1, i=\overline{0,n},$$

$$A_{i,k}^1=\left\langle \phi_k''-\phi_k,\phi_i\right\rangle, i=\overline{0,n-2}, k=\overline{2,n};\ A_{i,0}^1=-\left\langle 1,\phi_i\right\rangle, i=\overline{0,n-2}, k=0;\ A_{i,1}^1=-\left\langle \phi_1,\phi_i\right\rangle, i=\overline{0,n-2}, k=1,$$

$$f_i^1=\left\langle 1,\phi_i\right\rangle, i=\overline{0,n-2},\ f_{n-1}^1=f_n^1=1.$$

If the determinant of the matrix $A^1_{i,k};\ i,k=\overline{0,n}$ is nonzero, then the system of linear algebraic equations (22) has a unique solution $C^1=\left(A^1\right)^{-1}f^1$.

The second subproblem for solving problem (20) can be written in the form:

$$\begin{cases} u^2{}_{xx}+u^2{}_{yy}=0,\ 0<x<\pi, 0<y<\pi, \\ u^2\big|_{x=0}=u^2\big|_{x=\pi}=\sin y,\ 0\le x\le\pi, \\ u^2\big|_{y=0}=u^2\big|_{y=\pi}=0,\ 0\le y\le\pi, \end{cases} \tag{24}$$

$$u^2(x,y)=\sin y\cdot\sum_{k=0}^{n}C_k^2\phi_k(x)$$

with the boundary conditions

$$x=0:\sum_{k=0}^{n}C_k^2\left(-1\right)^k=1,\quad x=\pi:\sum_{k=0}^{n}C_k^2=1.$$

Substituting the solution $u^2(x, y)$ into the Laplace equation, we obtain:

$$\sin y\cdot\sum_{k=2}^{n}C_k^2\phi_k''(x)-\sin y\cdot\sum_{k=0}^{n}C_k^2\phi_k(x)=0\Leftrightarrow -C_0^2-C_1^2\phi_1+\sum_{k=2}^{n}C_k^2\left(\phi_k''-\phi_k\right)=0,$$

taking the scalar product of the last equation with the basis function $\phi_i(x), i=\overline{0,n-2}$, we obtain:

$$-C_0^2\left\langle 1,\phi_i\right\rangle-C_1^2\left\langle\phi_1,\phi_i\right\rangle+\sum_{k=2}^{n}C_k^2\left\langle\phi_k''-\phi_k,\phi_i\right\rangle=0, i=\overline{0,n-2}\Leftrightarrow\sum_{k=0}^{n}A_{i,k}^2C_k^2=f_i^2, i=\overline{0,n},$$

$$A_{i,k}^2=\left\langle\phi_k''-\phi_k,\phi_i\right\rangle, i=\overline{0,n-2}, k=\overline{2,n};\ A_{i,0}^2=-\left\langle 1,\phi_i\right\rangle, i=\overline{0,n-2}, k=0;\ A_{i,1}^2=-\left\langle\phi_1,\phi_i\right\rangle, i=\overline{0,n-2}, k=1, \tag{25}$$

$$f_i^2=0, i=\overline{0,n-2},\ f_{n-1}^2=f_n^2=1.$$

The vector of expansion coefficients is determined by the formula $C^2=\left(A^2\right)^{-1}f^2$.

Finally, the numerical solution of problem (20) can be represented as the sum of the solutions of problems (22) and (24):

$$u(x,y)=u^1(x,y)+u^2(x,y)=\sin x\cdot\sum_{k=0}^{n}C_k^1\phi_k(y)+\sin y\cdot\sum_{k=0}^{n}C_k^2\phi_k(x). \tag{26}$$

Table 1

Results of the program execution with the number of basis functions equal to 14 (n = 13)

x_j	y_i	$u^{exact}(x_j, y_i)$	$u^{num}(x_j, y_i)$
0.00000000000E+000	0.00000000000E+000	0.000000000E+000	0.00000000E+000
0.785398163397448	0.000000000000E+000	0.707106781186547	0.707106781188
1.57079632679490	0.000000000000E+000	1.00000000000000	1.000000000002
2.35619449019234	0.000000000000E+000	0.707106781186548	0.707106781188
3.14159265358979	0.000000000000E+000	1.22460635382-016	1.224606353E-016
0.392699081698724	0.392699081698724	0.430835084394819	0.4308350843945
1.17809724509617	0.392699081698724	0.549883843264189	0.549883843263
1.96349540849362	0.392699081698724	0.549883843264189	0.549883843263
2.74889357189107	0.392699081698724	0.430835084394819	0.4308350843945
0.000000000E+000	0.78539816339744	0.70710678118654	0.707106781188
0.785398163397448	0.785398163397448	0.412749869982035	0.412749869984
1.57079632679490	0.785398163397448	0.337619060675814	0.337619060678
2.35619449019234	0.785398163397448	0.412749869982035	0.412749869984
3.14159265358979	0.785398163397448	0.707106781186547	0.707106781188
0.392699081698724	1.17809724509617	0.600835882782445	0.600835882781
1.17809724509617	1.17809724509617	0.266992276932959	0.266992276929
1.96349540849362	1.17809724509617	0.266992276932959	0.266992276929
2.74889357189107	1.17809724509617	0.600835882782445	0.600835882781
0.00000000000E+000	1.57079632679490	1.00000000000000	1.000000000002

End of Table 1

x_j	y_i	$u^{exact}(x_j, y_i)$	$u^{num}(x_j, y_i)$
0.785398163397448	1.57079632679490	0.384414876168756	0.384414876171
1.57079632679490	1.57079632679490	0.195610446015160	0.195610446018
2.35619449019234	1.57079632679490	0.384414876168756	0.384414876171
3.14159265358979	1.57079632679490	1.00000000000000	1.0000000000028
0.392699081698724	1.96349540849362	0.600835882782446	0.600835882781
1.17809724509617	1.96349540849362	0.266992276932960	0.266992276929
1.96349540849362	1.96349540849362	0.266992276932960	0.266992276929
2.74889357189107	1.96349540849362	0.600835882782445	0.600835882781
0.00000000000E+000	2.35619449019234	0.707106781186548	0.707106781188
0.785398163397448	2.35619449019234	0.412749869982035	0.412749869984
1.57079632679490	2.35619449019234	0.337619060675815	0.337619060678
2.35619449019234	2.35619449019234	0.412749869982035	0.412749869984
3.14159265358979	2.35619449019234	0.707106781186548	0.707106781188
0.392699081698724	2.74889357189107	0.430835084394819	0.4308350843945
1.17809724509617	2.74889357189107	0.549883843264188	0.549883843263
1.96349540849362	2.74889357189107	0.549883843264188	0.549883843263
2.74889357189107	2.74889357189107	0.430835084394819	0.4308350843945
0.0000000000000E+000	3.14159265358979	1.2246063538E–016	1.224606353E–016
0.785398163397448	3.14159265358979	0.707106781186548	0.707106781188
1.57079632679490	3.14159265358979	1.00000000000000	1.00000000000272
2.35619449019234	3.14159265358979	0.707106781186548	0.707106781188
3.14159265358979	3.14159265358979	2.4492127076E–016	2.4492127076E–016

In the first and second columns of the table, the coordinates of the nodal points of the rectangle are given. The third and fourth columns present the exact solution (according to formula (21)) and the numerical solution (according to formula (26)) of problem (20) on a uniform grid $x_j = a + jh_1, y_i = c + ih_2, j = \overline{0, n_1}, i = \overline{0, n_2}, h_1 = \dfrac{b-a}{n_1}, h_2 = \dfrac{d-c}{n_2}, n_1 = n_2 = 40$. In the space of discrete functions with the uniform norm, the residual of the solution of Example (20) is equal to $\left\| u^{exact} - u^{num} \right\|_C = \max\limits_{j=\overline{0,n_1},\, i=\overline{0,n_2}} \left| u_{i.j}{}^{exact} - u_{i,j}{}^{num} \right| = 3.764766276503906\text{E}-012.$

In [14], the Poisson equation on a rectangular domain is approximated with eighth-order accuracy. The resulting finite-difference equation in the boundary value problem (20) is solved using the sweep method (Thomas algorithm) with a pentadiagonal matrix and by the simple iteration method. Using the program with analogous parameters $n_1 = 40 (n = 80)$; $n_2 = 40$ ($n_1 = 40$), ($m = 20000$) (the parameters used in [14] are given in parentheses), we obtain

$$\left\| u^{exact} - u^{num} \right\|_C = \max_{j=\overline{0,n_1},\, i=\overline{0,n_2}} \left| u_{i.j}{}^{exact} - u_{i,j}{}^{num} \right| = 7.355227538141662\text{E}-013.$$

In the Fortran programming language, using the DFIMSL library, it is possible to record the start time of the program execution (call cpu_time(t1)) and the end time (call cpu_time(t2)). Using the program from [14], the execution time is $t_2 - t_1 = 2.125$ seconds. The execution time of the program implementing algorithm (19)–(26) in the present work is $t_2 - t_1 = 7.8125\text{E} - 002$, which is 27.2 times (i. e., several tens of times) smaller than that reported in [14], while maintaining comparable solution accuracy in the uniform metric.

Let us consider the homogeneous Dirichlet boundary value problem (27) on a rectangular domain for the Poisson equation

$$\begin{cases} u_{xx} + u_{yy} = \sin x \cdot \sin y = f_1(x) f_2(y), f_1(0) = f_1(\pi) = 0, 0 < x < \pi, 0 < y < \pi, \\ u\big|_{x=0} = u\big|_{x=\pi} = 0,\ 0 \le x \le \pi, \\ u\big|_{y=0} = u\big|_{y=\pi} = 0,\ 0 \le y \le \pi \end{cases} \tag{27}$$

with the exact solution

$$u(x, y) = \frac{-\sin x \sin y}{2} : u_{xx} + u_{yy} = \frac{\sin x \sin y}{2} + \frac{\sin x \sin y}{2} = \sin x \sin y,$$

$$u(x,y)=f_1(x)\sum_{k=0}^{n}C_k\phi_k(y)=\sin x\cdot\sum_{k=0}^{n}C_k\phi_k(y)$$

and the boundary conditions

$$u(x,c)=f_1(x)\sum_{k=0}^{n}C_k\phi_k(c)=0\Leftrightarrow\sum_{k=0}^{n}C_k\left(-1\right)^k=0,\quad u(x,d)=f_1(x)\sum_{k=0}^{n}C_k\phi_k(d)=0\Leftrightarrow\sum_{k=0}^{n}C_k^1=0,$$

$$u(0,y)=\sin 0\cdot\sum_{k=0}^{n}C_k\phi_k(y)=0,u(\pi,y)=\sin\pi\cdot\sum_{k=0}^{n}C_k\phi_k(y)=0.$$

Substituting the solution $u(x, y)$ into the Laplace equation, we obtain:

$$f_1^{"}(x)\cdot\sum_{k=0}^{n}C_k\phi_k(y)+f_1(x)\cdot\sum_{k=2}^{n}C_k\phi_k^{"}(y)=f_1(x)f_2(y).$$

Taking the scalar product of the last equation with the basis function $\phi_i(y), i=\overline{0,n-2}$, in the general case we obtain:

$$\sum_{k=0}^{n}C_k\left\langle f_1^{"}(x)\phi_k(y),\phi_i(y)\right\rangle+\sum_{k=2}^{n}C_k\left\langle f_1(x)\phi_k^{"}(y),\phi_i(y)\right\rangle=\left\langle f_1(x)f_2(y),\phi_i(y)\right\rangle\Leftrightarrow$$

$$C_0\left\langle f_1^{"}(x),\phi_i\right\rangle+C_1\left\langle\phi_1 f_1^{"}(x),\phi_i\right\rangle+\sum_{k=2}^{n}C_k\left\langle f_1(x)\phi_k^{"}+\phi_k f_1^{"}(x),\phi_i\right\rangle=\left\langle f_1(x)f_2(y),\phi_i(y)\right\rangle, i=\overline{0,n-2}\Leftrightarrow \tag{28}$$

$$\Leftrightarrow\sum_{k=0}^{n}A_{i,k}C_k=f_i, i=\overline{0,n},$$

$$\begin{cases}A_{i,k}=\left\langle f_1(x)\phi_k^{"}(y)+\phi_k(y)f_1^{"}(x),\phi_i(y)\right\rangle, i=\overline{0,n-2}, k=\overline{2,n},\\ A_{i,0}=\left\langle f_1^{"}(x),\phi_i(y)\right\rangle, i=\overline{0,n-2}, k=0;\ A_{i,1}=\left\langle\phi_1(y)f_1^{"}(x),\phi_i(y)\right\rangle, i=\overline{0,n-2}, k=1,\\ A_{n-1,k}=\left(-1\right)^k, A_{n,k}=1, k=\overline{0,n},\\ f_i=\left\langle f_1(x)f_2(y),\phi_i(y)\right\rangle, i=\overline{0,n-2},\ f_{n-1}=f_n=0.\end{cases}$$

For example (27), equation (28) yields:

$$-\sin x\cdot\sum_{k=0}^{n}C_k\phi_k(y)+\sin x\cdot\sum_{k=2}^{n}C_k\phi_k^{"}(y)=\sin x\sin y\Leftrightarrow -C_0-C_1\phi_1+\sum_{k=2}^{n}C_k\left(\phi_k^{"}-\phi_k\right)=f_2(y)=\sin y,$$

$$-C_0\left\langle 1,\phi_i\right\rangle-C_1\left\langle\phi_1,\phi_i\right\rangle+\sum_{k=2}^{n}C_k\left\langle\phi_k^{"}-\phi_k,\phi_i\right\rangle=\left\langle\sin y,\phi_i\right\rangle, i=\overline{0,n-2}\Leftrightarrow\sum_{k=0}^{n}A_{i,k}C_k=f_i, i=\overline{0,n},$$

$$\sum_{k=0}^{n}A_{i,k}C_k=f_i, i=\overline{0,n}, \tag{29}$$

$$A_{i,k}=\left\langle\phi_k^{"}-\phi_k,\phi_i\right\rangle, i=\overline{0,n-2}, k=\overline{2,n};\ A_{i,0}=-\left\langle 1,\phi_i\right\rangle, i=\overline{0,n-2}, k=0;\ A_{i,1}=-\left\langle\phi_1,\phi_i\right\rangle, i=\overline{0,n-2}, k=1,$$

$$A_{n-1,k}=\left(-1\right)^k, A_{n,k}=1, k=\overline{0,n},\ f_i=\left\langle\sin y,\phi_i\right\rangle, i=\overline{0,n-2},\ f_{n-1}=f_n=0.$$

The system of linear algebraic equations (27) has a unique solution $u(x,y)=\sin x\cdot\sum_{k=0}^{n}C_k\phi_k(y),\ C=\left(A\right)^{-1}f$.

Table 2

Results of the program execution for Example (27) with 14 basis functions

x_j	y_i	$u^{exact}(x_j, y_i)$	$u^{num}(x_j, y_i)$
0.00000000000E+000	0.00000000000E+000	0.00000000000E+000	0.00000000000E+000
0.628318530717959	0.000000000000E+000	0.000000000000E+000	–2.8151350900E–013
1.25663706143592	0.000000000000E+000	0.000000000000E+000	–4.5549842585E–013
1.88495559215388	0.000000000000E+000	0.000000000000E+000	–4.5549842585E–013
2.51327412287183	0.0000000000000E+000	0.000000000000E+000	–2.8151350900E–013
3.14159265358979	0.000000000000E+000	0.000000000000E+000	–5.8651221762E–029
0.00000000000E+000	0.628318530717959	0.00000000000E+000	0.0000000000E+000
0.628318530717959	0.628318530717959	–0.172745751406263	–0.172745751406184

End of Table 2

x_j	y_i	$u^{exact}(x_j, y_i)$	$u^{num}(x_j, y_i)$
1.25663706143592	0.628318530717959	–0.279508497187474	–0.279508497187345
1.88495559215388	0.628318530717959	–0.279508497187474	–0.279508497187345
2.51327412287183	0.628318530717959	–0.172745751406263	–0.172745751406184
3.14159265358979	0.628318530717959	–3.5990277732E–017	–3.5990277732E–017
0.00000000000E+000	1.25663706143592	0.00000000000E+000	0.00000000000E+000
0.628318530717959	1.25663706143592	–0.279508497187474	–0.279508497188230
1.25663706143592	1.25663706143592	–0.452254248593737	–0.452254248594961
1.88495559215388	1.25663706143592	–0.452254248593737	–0.452254248594961
2.51327412287183	1.25663706143592	–0.279508497187474	–0.279508497188230
3.14159265358979	1.25663706143592	–5.8233492634E–017	–5.8233492635E–017
0.00000000000E+000	1.88495559215388	0.00000000000E+000	0.00000000000E+000
0.628318530717959	1.88495559215388	–0.279508497187474	–0.279508497188234
1.25663706143592	1.88495559215388	–0.452254248593737	–0.452254248594967
1.88495559215388	1.88495559215388	–0.452254248593737	–0.452254248594967
2.51327412287183	1.88495559215388	–0.279508497187474	–0.279508497188234
3.14159265358979	1.88495559215388	–5.8233492634E–017	–5.8233492635E–017
0.00000000000E+000	2.51327412287183	0.00000000000E+000	0.0000000000E+000
0.628318530717959	2.51327412287183	–0.172745751406263	–0.172745751406185
1.25663706143592	2.51327412287183	–0.279508497187474	–0.279508497187347
1.88495559215388	2.51327412287183	–0.279508497187474	–0.279508497187347
2.51327412287183	2.51327412287183	–0.172745751406263	–0.172745751406185
3.14159265358979	2.51327412287183	–3.5990277732E–017	–3.5990277732E–017
0.00000000000E+000	3.14159265358979	0.00000000000E+000	0.0000000000E+000
0.628318530717959	3.14159265358979	–3.59902777320E–017	–2.8152623432E–013
1.25663706143592	3.14159265358979	–5.8233492634E–017	–4.5551901586E–013
1.88495559215388	3.14159265358979	–5.8233492634E–017	–4.5551901586E–013
2.51327412287183	3.14159265358979	–3.59902777320E–017	–2.8152623432E–013
3.14159265358979	3.14159265358979	–7.49830360911E–033	–5.8653872988E–029

The program for Example (27) with the Poisson equation, taking into account formulas (28) and (29), yields a uniform error norm of $\left\|u^{exact}-u^{num}\right\|_C = \max\limits_{j=0,n_1,\, i=0,n_2}\left|u_{i.j}{}^{exact}-u_{i,j}{}^{num}\right| = 1.650901637617608\mathrm{E}-012,\ n_1=n_2=30, n=13$ with an execution time of $t_2 - t_1$ = 3.125000000000000E–002 seconds in Debug mode.

The third example differs from the second one by the factor x^2 instead of the function sin x:

$$\begin{cases} u_{xx}+u_{yy}=x^2\cdot\sin y,\ 0<x<\pi, 0<y<\pi, \\ u\big|_{x=0}=u\big|_{x=\pi}=0,\ 0\le x\le\pi, \\ u\big|_{y=0}=u\big|_{y=\pi}=0,\ 0\le y\le\pi. \end{cases} \tag{30}$$

The solution of Example (30) is sought in the form:

$$u(x,y)=f(x)\sin y, u_{xx}=f''\sin y, u_{yy}=-f(x)\sin(y),$$

$$u_{xx}+u_{yy}=\sin y\left(f''-f\right)=x^2\sin y \Leftrightarrow f''-f=x^2.$$

As a result, we obtain the boundary value problem

$$\begin{cases} f''-f=x^2,\ 0<x<\pi, \\ f(0)=f(\pi)=0. \end{cases}$$

The general solution of the homogeneous equation $f''-f=0$ has the form $f_{oo}(x)=A\,\mathrm{ch}\,x+B\,\mathrm{sh}\,x$, while the particular solution of the nonhomogeneous equation is sought in the form $f_ч(x)=C+Dx^2 \Leftrightarrow 2D-C-Dx^2=x^2 \Leftrightarrow$ $D=-1, C=2D=-2; f_ч(x)=-x^2-2$.

Thus, the general solution of the nonhomogeneous equation is:

$$f_{он}(x) = A\operatorname{ch} x + B\operatorname{sh} x - \cos x/2;\ f_{он}(0) = 0 \Leftrightarrow A = 1/2;\ f_{он}(\pi) = 0 \Leftrightarrow \operatorname{ch}\pi/2 + B\operatorname{sh}\pi + 1/2 = 0 \Leftrightarrow$$

$$B = \frac{-(1+\operatorname{ch}\pi)}{2\operatorname{sh}\pi},\ f(x) = \frac{\operatorname{ch} x}{2} - \left(\frac{(1+\operatorname{ch}\pi)}{2\operatorname{sh}\pi}\right)\operatorname{sh} x - \frac{\cos x}{2}.$$

Finally, the exact solution of Example (30) (presented in the third column of Table 3) has the form:

$$u(x,y) = \left(2\operatorname{ch} x + \left(\frac{2(1-\operatorname{ch}\pi)+\pi^2}{\operatorname{sh}\pi}\right)\operatorname{sh} x - x^2 - 2\right)\sin y.$$

Table 3

Results of the program execution for Example (30) using algorithm (28) with 14 basis functions

x_j	y_i	$u^{exact}(x_j, y_i)$	$u^{num}(x_j, y_i)$
0.00000000E+000	0.00000000000E+000	0.00000000000E+000	0.00000000000E+000
0.628318530717959	0.00000000000E+000	0.0000000000E+000	0.0000000000E+000
1.25663706143592	0.00000000000E+000	0.000000000000E+000	0.00000000000E+000
1.88495559215388	0.00000000000E+000	0.000000000000E+000	0.000000000000E+000
2.51327412287183	0.000000000000E+000	0.000000000000E+000	0.000000000000E+000
3.14159265358979	0.000000000000E+000	0.000000000000E+000	0.000000000000E+000
0.00000000000E+000	0.628318530717959	0.00000000000E+000	–4.4197281494E–012
0.628318530717959	0.628318530717959	–0.378365183428067	–0.378365183428139
1.25663706143592	0.628318530717959	–0.800948970019281	–0.800948970022709
1.88495559215388	0.628318530717959	–1.15615367583243	–1.15615367583571
2.51327412287183	0.628318530717959	–1.11556919336887	–1.11556919336957
3.14159265358979	0.628318530717959	0.000000000000E+000	5.937775557832E–012
0.00000000000E+000	1.25663706143592	0.00000000000E+000	–7.15127036687E–012
0.628318530717959	1.25663706143592	–0.612207726946317	–0.612207726946200
1.25663706143592	1.25663706143592	–1.29596265674542	–1.29596265675096
1.88495559215388	1.25663706143592	–1.87069594371500	–1.87069594372030
2.51327412287183	1.25663706143592	–1.80502887167314	–1.80502887167427
3.14159265358979	1.25663706143592	0.000000000000E+000	9.607522670141E–012
0.00000000000E+000	1.88495559215388	0.00000000000E+000	–7.1512703668E–012
0.628318530717959	1.88495559215388	–0.612207726946200	–0.612207726946317
1.25663706143592	1.88495559215388	–1.29596265674542	–1.29596265675096
1.88495559215388	1.88495559215388	–1.87069594371500	–1.87069594372030
2.51327412287183	1.88495559215388	–1.80502887167314	–1.80502887167427
3.14159265358979	1.88495559215388	0.000000000000E+000	9.607522670141E–012
0.00000000000E+000	2.51327412287183	0.00000000000E+000	–4.4197281494E–012
0.628318530717959	2.51327412287183	–0.378365183428067	–0.378365183428139
1.25663706143592	2.51327412287183	–0.800948970019281	–0.800948970022709
1.88495559215388	2.51327412287183	–1.15615367583243	–1.15615367583571
2.51327412287183	2.51327412287183	–1.11556919336887	–1.11556919336957
3.14159265358979	2.51327412287183	0.00000000000E+000	5.937775557832E–012
0.00000000000E+000	3.14159265358979	0.00000000000E+000	–9.20817110142E–028
0.628318530717959	3.14159265358979	–7.88295395102E–017	–7.88295395102E–017
1.25663706143592	3.14159265358979	–1.66871692331E–016	–1.66871692331E–016
1.88495559215388	3.14159265358979	–2.40875920567E–016	–2.40875920568E–016
2.51327412287183	3.14159265358979	–2.32420448964E–016	–2.32420448965E–016
3.14159265358979	3.14159265358979	0.000000000000E+000	1.23709086734E–027

The program for Example (30) with the Poisson equation, taking into account formulas (28) and (29), yields a uniform error norm of $\left\|u^{exact} - u^{num}\right\|_C = \max\limits_{j=0,n_1,\, i=0,n_2}\left|u_{i,j}{}^{exact} - u_{i,j}{}^{num}\right| = 1.010194715616629\text{E}-011,\ n_1 = n_2 = 70, n = 13$ with an execution time of $t_2 - t_1$ = 1.562500000000000E–002 seconds in Debug mode.

Let us consider the fourth example:

$$\begin{cases} u_{xx} + u_{yy} = \sin(x+y) = \sin x \cos y + \cos x \sin y,\ 0 < x < \pi, 0 < y < \pi, \\ u\big|_{x=0} = u\big|_{x=\pi} = 0,\ 0 \le x \le \pi, \\ u\big|_{y=0} = u\big|_{y=\pi} = 0,\ 0 \le y \le \pi. \end{cases} \tag{31}$$

Due to the symmetry of Example (31), a simpler problem can be solved:

$$\begin{cases} u_{xx} + u_{yy} = \cos x \sin y,\ 0 < x < \pi, 0 < y < \pi, \\ u\big|_{x=0} = u\big|_{x=\pi} = 0,\ 0 \le x \le \pi, \\ u\big|_{y=0} = u\big|_{y=\pi} = 0,\ 0 \le y \le \pi. \end{cases} \tag{32}$$

The solution of Example (32) is sought in the form

$$u(x,y) = f(x)\sin y, u_{xx} = f'' \sin y, u_{yy} = -f(x)\sin(y),$$

$$u_{xx} + u_{yy} = \sin y\left(f'' - f\right) = \cos x \sin y \Leftrightarrow f'' - f = \cos x.$$

As a result, we obtain the boundary value problem

$$\begin{cases} f'' - f = \cos x,\ 0 < x < \pi, \\ f(0) = f(\pi) = 0. \end{cases}$$

The general solution of the homogeneous equation $f'' - f = 0$ has the form $f_{oo}(x) = A\,\mathrm{ch}\,x + B\,\mathrm{sh}\,x$, while the particular solution of the nonhomogeneous equation is sought in the form $f_ч(x) = C\cos x \Leftrightarrow -2C\cos x = \cos x \Leftrightarrow C = -1/2$; $f_ч(x) = -\cos x / 2$.

Thus, the general solution of the nonhomogeneous equation can be written as:

$$f_{он}(x) = A\,\mathrm{ch}\,x + B\,\mathrm{sh}\,x - \cos x/2;\ f_{он}(0) = 0 \Leftrightarrow A = 1/2;\ f_{он}(\pi) = 0 \Leftrightarrow \mathrm{ch}\,\pi/2 + B\,\mathrm{sh}\,\pi + 1/2 = 0 \Leftrightarrow$$

$$B = \frac{-(1+\mathrm{ch}\,\pi)}{2\,\mathrm{sh}\,\pi},\ f(x) = \frac{\mathrm{ch}\,x}{2} - \left(\frac{(1+\mathrm{ch}\,\pi)}{2\,\mathrm{sh}\,\pi}\right)\mathrm{sh}\,x - \frac{\cos x}{2}.$$

Finally, the exact solution of Example (32) has the form:

$$u(x,y) = \left(\frac{\mathrm{ch}\,x}{2} - \left(\frac{(1+\mathrm{ch}\,\pi)}{2\,\mathrm{sh}\,\pi}\right)\mathrm{sh}\,x - \frac{\cos x}{2}\right)\sin y.$$

The exact solution of Example (31) is then obtained taking into account the symmetry:

$$u(x,y) = \left(\frac{\mathrm{ch}\,x}{2} - \left(\frac{(1+\mathrm{ch}\,\pi)}{2\,\mathrm{sh}\,\pi}\right)\mathrm{sh}\,x - \frac{\cos x}{2}\right)\sin y + \left(\frac{\mathrm{ch}\,y}{2} - \left(\frac{(1+\mathrm{ch}\,\pi)}{2\,\mathrm{sh}\,\pi}\right)\mathrm{sh}\,y - \frac{\cos y}{2}\right)\sin x. \tag{33}$$

The exact solution computed by formula (33) at the nodes of the uniform grid is presented in the third column of Table 4, while the numerical solution is given in the fourth column.

Table 4

Results of the program execution for Example (31) with 14 basis functions

x_j	y_i	$u^{exact}(x_j, y_i)$	$u^{num}(x_j, y_i)$
0.00000000000E+000	0.00000000000E+000	0.00000000000E+000	0.00000000000E+000
0.628318530717959	0.00000000000E+000	0.00000000000E+000	–1.0441573502E–013
1.25663706143592	0.000000000000E+000	0.000000000000E+000	–1.6894820822E–013
1.88495559215388	0.000000000000E+000	0.000000000000E+000	–1.6894820822E–013
2.51327412287183	0.000000000000E+000	0.000000000000E+000	–1.0441573502E–013
3.14159265358979	0.000000000000E+000	0.000000000000E+000	–2.1754232868E–029
0.0000000000E+000	0.628318530717959	0.00000000000E+000	–1.0441573502E–013
0.628318530717959	0.628318530717959	–0.197551489645173	–0.197551489645269
1.25663706143592	0.628318530717959	–0.209856683772612	–0.209856683772669
1.88495559215388	0.628318530717959	–0.109788341001450	–0.109788341001551
2.51327412287183	0.628318530717959	2.220446049250E–016	1.42941214420E–015
3.14159265358979	0.628318530717959	5.014789931644E–016	1.04386956393E–013

End to Table 4

0.000000000000E+000	1.25663706143592	0.00000000000E+000	–1.6894820822E–013
0.628318530717959	1.25663706143592	–0.209856683772612	–0.209856683772669
1.25663706143592	1.25663706143592	–0.161913979801615	–0.161913979801545
1.88495559215388	1.25663706143592	2.636779683484E–016	–1.6792123247E–015
2.51327412287183	1.25663706143592	0.109788341001451	0.109788341001552
3.14159265358979	1.25663706143592	8.342836304895E–016	1.68924517002E–013
0.000000000000E+000	1.88495559215388	0.000000000000E+000	–1.6894820822E–013
0.628318530717959	1.88495559215388	–0.109788341001450	–0.109788341001551
1.25663706143592	1.88495559215388	2.636779683484E–016	–1.6653345369E–015
1.88495559215388	1.88495559215388	0.161913979801615	0.161913979801542
2.51327412287183	1.88495559215388	0.209856683772613	0.209856683772670
3.14159265358979	1.88495559215388	8.551321168875E–016	1.68945365488E–013
0.00000000000E+000	2.51327412287183	0.00000000000E+000	–1.0441573502E–013
0.628318530717959	2.51327412287183	2.081668171172E–016	1.41553435639E–015
1.25663706143592	2.51327412287183	0.109788341001451	0.109788341001552
1.88495559215388	2.51327412287183	0.209856683772613	0.209856683772670
2.51327412287183	2.51327412287183	0.197551489645174	0.197551489645272
3.14159265358979	2.51327412287183	5.426373598439E–016	1.04428114760E–013
0.00000000000E+000	3.14159265358979	0.00000000000E+000	–2.1754232868E–029
0.628318530717959	3.14159265358979	5.014789931644E–016	1.04386956393E–013
1.25663706143592	3.14159265358979	8.342836304895E–016	1.68924517002E–013
1.88495559215388	3.14159265358979	8.551321168875E–016	1.68945365488E–013
2.51327412287183	3.14159265358979	5.426373598439E–016	1.04428114760E–013
3.14159265358979	3.14159265358979	2.175337872185E–031	4.35050491505E–029

The program for Example (30) with the Poisson equation, taking into account formulas (28) and (29), yields a uniform error norm of $\left\|u^{exact}-u^{num}\right\|_C = \max\limits_{j=0,n_1,\, i=0,n_2}\left|u_{i.j}{}^{exact}-u_{i,j}{}^{num}\right| = 7.962103198977388\text{E}-013,\ n_1=n_2=60, n=13$ with an execution time of $t_2 - t_1$ = 1.562500000000000E–002 seconds in Release mode.

Discussion

1. In this work, an algorithm for solving boundary value problems by the modified Bubnov-Galerkin method (3)–(18) is proposed for the first time. The method is applied to linear homogeneous partial differential equations of elliptic type on a rectangle (1) with zero values of the unknown function at the rectangle vertices. The coefficients $g_i(x, y)$ at the partial derivatives may depend on the spatial coordinates but do not depend on the unknown function.

2. Algorithm (3)–(18) can be generalized to solve boundary value problems with an arbitrary linear partial differential equation on a rectangle. In this case, the sum for the differential operator with a single index $\sum\limits_{j=0}^{m} g_j(x,y)\dfrac{\partial^j\partial^{m-j}}{\partial x^j\partial y^{m-j}}$ should be replaced by a sum with two indices $\sum\limits_{i,j=0}^{m_1,m_2} g_{i,j}(x,y)\dfrac{\partial^i\partial^j}{\partial x^i\partial y^j}$. The condition of zero values of the function at the vertices of the rectangle remains unchanged.

3. The paper considers four examples of solving a boundary value problem with the Laplace (Poisson) equation on a rectangle (19)–(26). In addition, the Laplace equation on a rectangle with homogeneous boundary conditions in the general form (27), (28) and with a nonhomogeneous right-hand side with separable variables is solved using the Bubnov-Galerkin method with zero values of the solution at the rectangle vertices.

4. The accuracy of the solution of the boundary value problem for the Poisson equation obtained by the modified Bubnov-Galerkin method is comparable to the accuracy achieved by the pentadiagonal sweep method, while the computation time is approximately thirty times smaller.

5. Each nonhomogeneous boundary condition on one of the four sides of the square generates a separate auxiliary boundary value problem. If the boundary conditions on opposite sides of the square coincide (symmetry of boundary conditions), the problem can be reduced to two simpler boundary value problems.

6. As a system of linearly independent coordinate functions in the boundary value problem, power-type functions (3), (4) were used. These functions take the value −1 at the left endpoint of the interval and +1 at the right endpoint (for functions with odd indices), and the value +1 at both endpoints of the interval (for functions with even indices).

7. A weight matrix of the quadrature formula on a rectangle with twelfth-order accuracy is proposed for computing the scalar product of two functions.

8. The last two equations in each system of linear algebraic equations represent the conditions defining the set of admissible functions in the boundary value problem on a rectangle.

Conclusion. Four examples of the Poisson equation on a rectangular domain, in which the unknown function is zero at the rectangle vertices, were numerically solved using algorithm (28), (29). In addition, algorithm (5)–(19) was developed for solving boundary value problems with homogeneous partial differential equations of elliptic type of arbitrary order. The results demonstrate the successful application of the modified Bubnov-Galerkin method for solving boundary value problems on rectangular domains.

References

1. Bahvalov N.S., Zhidkov N.P., Kobelkov G.M. *Numerical methods: a textbook for students of physics and mathematics specialties of higher educational institutions*. Moscow: BINOM. Knowledge laboratory; 2011. 636 p. (In Russ.)

2. Bahvalov N.S., Lapin A.V., Chizhonkov E.V. *Numerical methods in problems and exercises*. Moscow: BINOM. Knowledge laboratory; 2010. 240 p. (In Russ.)

3. Pikulin V.P., Pokhozhaev S.I. *Practical course on the equations of mathematical physics*. Moscow: MTSNMO; 2004. 208 p. (In Russ.)

4. Sukhinov A.I., Protsenko E.A. Parallel numerical implementation of wave hydrodynamics models taking into account the features of vertical turbulent exchange. *Mathematical Modeling*. 2023;35(12):51–68. (In Russ.) https://doi.org/10.20948/mm-2023-12-04

5. Sukhinov A.I., Chistyakov A.E., Nikitina A.V., Atayan A.M., Litvinov V.N. Method for solving grid equations for hydrodynamic problems in flat areas. *Mathematical Modeling*. 2023;35(3):35–58. (In Russ.) https://doi.org/10.20948/mm-2023-03-03

6. Sukhinov A.I., Chistyakov A.E., Sidoryakina V.V., Kuznetsova I.I., Atayan A.M. The use of parallel computing to assess the process of transport of pollutants in shallow water bodies. *News of Saratov University. New series. Series: Mathematics. Mechanics. Informatics*. 2024;24(2):298–315. (In Russ.) https://doi.org/10.18500/1816-9791-2024-24-2-298-315

7. Sidoryakina V.V., Sukhinov A.I. Construction and study of the proximity of solutions in L2 of two boundary value problems for the model of transport of multicomponent suspended matter in coastal systems. *Journal of Computational Mathematics and Mathematical Physics*. 2023;63(10):1721–1732. (In Russ.) https://doi.org/10.31857/S0044466923100149

8. Sidoryakina V.V., Solomaha D.A. Symmetrized versions of the Seidel and upper relaxation methods for solving two-dimensional difference problems of elliptic. *Computational Mathematics and Information Technologies*. 2023;7(3):12–19. (In Russ.) https://doi.org/10.23947/2587-8999-2023-7-3-12-19

9. Burova I.G., Altsybeev G.O. Application of second-order spline approximations to the solution of integral equations of the second kind. *Computational methods and programming*. 2025;26(2):175–191. (In Russ.) https://doi.org/10.26089/NumMet.v26r213

10. Chernyshev A.D., Goryainov V.V., Kuznetsov S.V., Nikiforova O.Yu. Application of fast expansions to construct exact solutions to the problem of deflection of a rectangular membrane under the action of a variable load. *Bulletin of Tomsk State University. Mathematics and mechanics*. 2021;70:127–142. (In Russ.) https://doi.org/10.17223/19988621/70/11

11. Volosova N.K., Volosov K.A., Volosova A.K., Pastukhov D.F., Pastukhov Yu.F. Modified Bubnov-Galerkin method for solving boundary value problems with a linear ordinary differential equation. *Computational Mathematics and Information Technologies*. 2024;8(3):23–33. (In Russ.) https://doi.org/10.23947/2587-8999-2024-8-3-23-33

12. Volosova N.K., Volosov K. A., Volosova A. K., Pastukhov D.F., Pastukhov Yu.F. Increasing the accuracy of solving boundary value problems with linear ordinary differential equations by the Bubnov-Galerkin method. *Computational Mathematics and Information Technologies*. 2024;8(4):7–18. (In Russ.) https://doi.org/10.23947/2587-8999-2024-8-4-7-18

13. Volosova N.K., Volosov K. A., Volosova A. K., Pastukhov D.F., Pastukhov Yu.F. Solving boundary value problems for some nonlinear differential equations by the Bubnov-Galerkin method. *Computational Mathematics and Information Technologies*. 2025; 9(1):7–19. (In Russ.) https://doi.org/10.23947/2587-8999-2025-9-1-7-19

14. Pastukhov D.F., Pastukhov Yu.F. Approximation of the Poisson equation on a rectangle of increased accuracy. *Bulletin of Polotsk State University. Series C. Fundamental sciences*. 2017;12:62–77. (In Russ.)

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A.K. Volosova: translation; study of the history of the task; literature.

N.A. Gurieva: examples.

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