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Review article

Grid-characteristic methods. 55 years of developing and solving complex dynamic problems

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Abstract

The development of a computational method is not a simple matter and boils down to replacing the differential operator with a difference one. To construct it, it is necessary to correctly set a mathematical problem that is adequate to the physical one under consideration. In addition, the algorithm must meet some other requirements. Therefore, to create a numerical algorithm requires not only ingenuity and imagination, but also a deep understanding of the reasons why these requirements are caused.

Systems of partial differential equations of hyperbolic type are used to describe the unsteady behavior of continuous media. To solve these problems, characteristic methods were developed in such a way as to take into account the corresponding properties of hyperbolic equations and to be able to build a so-called characteristic irregular grid adapting to the solution of the problem. Methods of end-to-end counting have been developed that take into account the properties of systems of hyperbolic equations — inverse methods of characteristics or grid-characteristic methods.

In grid-characteristic methods, a regular computational grid is used, not a solvable initial system is approximated on it, but compatibility conditions along characteristic lines with interpolation of the desired functions at the points of intersection of characteristics with a coordinate line on which the data is already known. The obtained characteristic form of the gas dynamics equations makes it possible to understand how to set the boundary conditions correctly.

It is necessary to take into account the physical side of the problem being solved, when developing the method. When developing the method, it is necessary to take into account the physical side of the problem being solved. At the same time, the method must meet certain requirements, the understanding of which is necessary during its development.

Keywords: numerical methods, systems of hyperbolic differential equations, wave processes, numerical solutions on characteristic grids, irregular grid, grid-characteristic methods

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Обзорная статья

Сеточно-характеристические методы. 55 лет разработки и решения сложных динамических задач

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Аннотация

Введение. Разработка вычислительного метода не является делом простым и сводящимся к замене дифференциального оператора разностным. Для его построения необходимо грамотно поставить математическую задачу, адекватную рассматриваемой физической. Кроме того, алгоритм должен удовлетворять и некоторым другим требованиям. Поэтому для создания численного алгоритма нужна не только изобретательность и фантазия, но и глубокое понимание причин, которыми эти требования вызываются.

Для описания нестационарного поведения сплошных сред используются системы дифференциальных уравнений в частных производных гиперболического типа. Для решения этих проблем характеристические методы разрабатывались таким образом, чтобы учесть соответствующие свойства гипер-болических уравнений и иметь возможность строить т.н. характеристическую, адаптирующую к решению задачи, нерегулярную сетку. Разработаны методы сквозного счета, учитывающие свойства систем уравнений гиперболического типа — обратные методы характеристик или сеточно-характеристические методы.
В сеточно-характеристических методах используется регулярная расчетная сетка, на ней аппроксимируется не решаемая исходная система, а условия совместимости вдоль характеристических линий с интерполяцией искомых функций в точках пересечения характеристик с координатной линией, на которой данные уже известны. Полученная характеристическая форма уравнений газовой динамики позволяет понять, как правильно ставить граничные условия.
Построение численного метода не является простым делом и не сводится к формальной замене производных аппроксимирующими их разностными соотношениями (например, с помощью конечных разностей). При разработке метода необходимо учитывать физическую сторону решаемой задачи. При этом метод должен удовлетворять определенным требованиям, понимание которых необходимо при его разработке.

Ключевые слова: численные методы, системы дифференциальных уравнений гиперболического типа, волновые процессы, численные решения на характеристических сетках, нерегулярная сетка, сеточно-характеристические методы.

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Introduction. Systems of partial differential equations of hyperbolic type are usually used to describe the unsteady behavior of continuous media — gas, solid deformable body, liquid, plasma. These are Euler systems in gas dynamics, Lame systems in elasticity theory, Timoshenko systems in shell theory, Maxwell systems in magnetic hydrodynamics, Bio systems in fluid-saturated porous media, Marchuk systems in climatology and oceanology, etc. The fields of application of such systems are extensive. The corresponding numerical methods used to solve these systems originate in the 40−50s of the XX century. Their development was connected, first of all, with the need to predict the consequences of a nuclear explosion (the consequences of the tragedy of Hiroshima and Nagasaki and the further implementation of the nuclear program in the Soviet Union, which was a necessary counterweight to the nuclear threat from overseas). Soon there were problems about the flow of blunted bodies in dense layers of the atmosphere moving at hypersonic speeds (the problem of delivery). The first difference schemes for solving problems of gas dynamics were created — Lax, Lax-Wendroff, Courant-Izakson-Rice (Godunov), Landau-Meiman-Khalatnikov, Rusanov, etc. A detailed description of the history of the schemes and their overview can be found in well-known works [1−13].

Systems of hyperbolic differential equations have the most general properties:

– the equations describe wave processes, propagation of weak perturbations, or wavefront;

– in the case of linear problems on the propagation of wave fronts, the characteristics can be found independently of the solution of the equation (or system of equations) under consideration, which makes it possible to obtain exact solutions of Dalembert, Kirchhoff, as well as numerical solutions on characteristic grids;

– in the case of nonlinear partial differential equations, the intersection of characteristics is possible when discontinuities occur;

– the characteristic properties of hyperbolic equations make it possible to study the correctness of the formulation of initial boundary value problems, for example, to determine the number of boundary conditions and conditions on the interface surfaces of media.

The main feature of hyperbolic equations or systems of differential equations is the finite velocity of propagation of waves (or perturbations) in the medium, as well as the presence of characteristic surfaces (lines — in the one-dimensional case) denoting the domain of dependence of solutions. On these surfaces, the number of independent variables decreases by one. For the first time, the characteristic properties of such systems were studied in [14], where the concept of Riemann invariants was introduced. Numerical methods that take into account the characteristic properties of hyperbolic systems of equations are described in detail in [2−12]. The important fact is also noted that using the method of characteristics, the theorems of existence, uniqueness and continuous dependence of the solution of the classical Cauchy problem on the input data were proved [1]. However, this domain is limited in the nonlinear case, because, unlike the linear one, these solutions can have, after some time, unlimited first derivatives — the so-called gradient catastrophe, i.e. discontinuities can arise from smooth initial data. In this case, they speak of a generalized solution of the equations of gas dynamics. A generalized solution, in this case, is understood as a solution that satisfies the laws of conservation of mass, momentum, energy, as well as an inequality that means an increase in entropy in a closed system. From a mathematical point of view, the requirement of increasing entropy guarantees the uniqueness of the generalized solution, as well as its stability with respect to small perturbations. It follows from what has been said that the construction of a numerical method is not a simple matter and is not reduced to the formal replacement of derivatives by approximating their difference relations (for example, using finite differences). When developing the method, it is necessary to take into account the physical side of the problem being solved. At the same time, the method must meet certain requirements, the understanding of which is necessary during its development.

Characteristic methods were developed in such a way as to take into account the corresponding properties of hyperbolic equations and to be able to build a so-called characteristic irregular grid adapting to the solution to solve these problems. These methods are called direct characterization methods [14−17]. Direct characteristic methods allow us to distinguish discontinuities, of which two types can be distinguished: in the first case, the structure of the solution and the location of the discontinuity are a priori known; in the second case, discontinuities occur over time. As for the first type of discontinuities, their isolation in the multidimensional case is a difficult task, which many researchers have been solving, for example [2, 7, 9]. In the second case, the numerical algorithm should detect gaps formed over time, after which it is possible to solve the problem of separation of the gap. The solution of such problems presents the greater difficulties, the more gaps in the field of integration. For this reason, methods of end-to-end counting have been developed that take into account the properties of systems of hyperbolic equations — inverse methods of characteristics or grid-characteristic methods (grid-characteristic method, GCM). These methods use a regular calculation grid. However, it approximates not the initial system to be solved, but the compatibility conditions along the characteristic lines with the interpolation of the desired functions at the points of intersection of the characteristics with the coordinate line on which the data is already known. In the multidimensional case — at the intersection points of the intersection lines of characteristic and coordinate planes with planes with known data. Works are devoted to the development of these methods [2, 7, 9, 18−22].

The first methods of the first order of accuracy were proposed [2, 9, 18, 23−25], then the second [25−27] and the third [27−30]. Subsequently , higher — order schemes were developed [19, 31−35, 52, 54, 57, 58].

In such approaches (through-counting methods), the approximation of derivatives through discontinuities is realized, which, when numerically solving the problem, have a so-called “blur” region, the value of which is determined by the numerical viscosity (dissipation) of the method used. The width of this zone decreases with increasing order of accuracy of the numerical method. In addition, when numerically solving problems with large gradients of the desired functions by methods having an approximation order higher than the first, numerical (non-physical) oscillations may appear. Different approaches are used to eliminate them (or reduce the amplitude). In the first of them, additional dissipative terms were used, in particular, artificial diffusion (or viscosity), both linear and quadratic, which was published in the works of Neumann and Rachtmayer [36]. Studies of the properties and modifications of such artificial solutions can be found in other works [6, 8, 36−38], etc. Generalization of such dissipative additives to the multidimensional case was considered in the review paper [39]. It is noted that artificial dissipative terms change the solution of the original problem [38], so the resulting numerical solution of the problem should be tested. In areas where large gradients are absent, methods of a higher order of accuracy (more than the first) can be used. The latter statement, as well as the monotonicity property of first-order approximation schemes, formed the basis of the idea of hybrid methods.

In the theory of difference schemes, an important concept of monotone (majorant) schemes or schemes with positive Friedrichs approximation is introduced. Such schemes preserve the monotonic character of the numerical solution (in the one-dimensional case) on any time layer, if this is the case in the exact solution of the problem. The use of non-monotonic difference schemes leads to the appearance of non-physical oscillations in the numerical solution (i. e., oscillations having a numerical origin). For a one — dimensional linear transfer equation of S. K. Godunov [42] proved the theorem that there are no explicit linear monotone schemes with an approximation order higher than the first. In [37], this theorem was extended to the case of an arbitrary template (for implicit or multilayer schemes).

To determine the monotony of the difference scheme, explicit linear two-layer schemes are presented in the following form:

$$v_m^{n+1} = \sum_i c_i\left(\tau, h\right) \cdot v_{m+i}^n,\ i = 0, \pm 1, ..,$$

where $n\tau = t^n$ (t is the time; τ is the time step; $n = 0,1...$);
$x_m = mh$ (x is the coordinate; h is the coordinate step, $m = 0,\pm1...$);
$v^n_m = v(t^n, x^m)$ is the desired grid function.

There are several definitions of monotony [22].

1. Schemes monotonous according to Friedrichs [41], for them: $c_i \geq 0$.

2. Schemes monotonous according to Godunov [42], for which the following inequalities are fulfilled: $v_m^{n+1} - v_m^n \geq 0$, by $v_{m+1}^n - v_m^n$ that is, on all time layers, coordinate one-sided differences do not change the sign.

3. Harten monotonic schemes [43]: $\sum_m \left|v_{m+1}^n - v_m^{n+1}\right| \leq \sum_m \left|v_{m+1}^n - v_m^n\right|$, where $TV\left(u_m^n\right) = \sum_m \left|v_{m+1}^n - v_m^{n+1}\right|$ there is a complite variation of the grid function.

4. Difference schemes based on the characteristic properties of the exact solution [19, 45] for which the inequality is fu lfilled: $\left\{v_1^n, v_2^n\right\} \leq v_m^{n+1} = v \leq \max\left\{v_1^n, v_2^n\right\}$;where v_1^n, v_2^n are the values of the grid function on the time layer t^n in the two brush nodes $\left\{t^{n+1}, x_m\right\}$ closest to the one originating from the node (minimum condition). It is shown that in the linear one-dimensional case, all the above definitions of monotonicity are equivalent and are a sufficient condition for the stability of difference schemes.

In the field of smooth numerical solutions, one can use difference schemes of an order of accuracy higher than the first, i. e., in accordance with Godunov's theorem [42], which are not monotonic. However, to eliminate (or reduce the amplitude), non-physical (numerical) oscillations in areas with large gradients of solutions, it is necessary to use monotonic schemes of the first order of approximation. The combination of these two contradictory requirements was realized in the idea of constructing hybrid difference schemes, which was first proposed by Fedorenko in [28]. These schemes are nonlinear, i. e. depending on the solution, and can be locally, at various points in the integration domain, change the order of approximation. Hybrid methods make it possible to implement end-to-end counting using schemes of an increased order of accuracy in areas with smooth solutions — in areas of large gradients of the numerical solution. This makes it possible to combine various positive qualities of difference schemes with different order of approximation in one computational algorithm. To clarify the end-to-end numerical solutions near the discontinuities in [46], it was recommended to use a differential shock wave analyzer, which allows localizing the discontinuity using the results of the end-to-end calculation and further refine the numerical solution. In [28], a method was also described for switching from a first-order scheme to a second-order scheme based on the ratio of the second or third based on the ratio of the second finite difference to the first. Hybrid schemes for linear and quasi-linear transfer equations with a smooth switch from one circuit to another were given in [47]. A hybrid scheme for a system of hyperbolic equations based on a combination of Lax [23] and Lax-Wendroff schemes [25] was proposed by Harten [48].

Van Lear's works [49, 50] also described a special algorithm for monotonizing the Lax-Wendroff scheme. Colgan in [51] proposed hybridization of the Godunov scheme using several templates, as well as a limiter (limiter) minmod. The first hybrid grid-characteristic difference schemes were described in the works of Kholodov and Petrov in [20], and their development in [11, 21, 35]. In [44], a hybrid method based on flow correction (flux corrected transport) was proposed, in which at the first stage a solution is obtained using a scheme of the first order of accuracy, at the second a term called “antidiffusion” is added, which allows increasing the order to the second.

The use of ideas of hybridity [28], correction of flows [44], limiters [49] led to the creation of TVD schemes (total variation diminishing), [43]. A review of the limiters for this class of hybrid methods is presented in [52].

Further development of TVD methods led to the emergence of new schemes: ENO [53], TVB [54], TVD2, UNO, UNO2 [54], WENO [55], WAF (TORO [56]). The emergence of these methods led to the creation of high-order accuracy schemes (high resolution schemes), see, for example, the monographs of Thoreau, Tolstykh [57, 58].

When numerically solving multidimensional dynamic problems, one often has to deal with moving boundaries, complex integration domains. For this purpose, mobile computational grids [59] and adaptive grids [60] are used. The theory and review of works on the construction of computational grids in complex integration domains are given in monographs [61, 62]. In cases where there is a dynamic expansion of parts of a continuous medium (the expansion of gas, liquid, plasma under dynamic influences, the destruction of deformable solids during explosions, impacts, etc.), the particle methods first proposed by Harlow [63], Belotserkovsky and Davydov in [64] (the method of large particles) are useful. Another approach to solving similar problems turned out to be the smooth particle method (SPH) [66, 67].

Grid-characteristic methods were further developed in [68] (method on unstructured tetrahedral grids), [69] (combined method: SPH and grid-characteristic), [34] (methods of increased order of approximation). The class of compact schemes that allow constructing schemes of an increased order of accuracy on compact templates is developed in [58, 70–72], and in [71, 72] the grid-characteristic method was used for their construction. The discontinuous Galerkin method [73, 74], combining the capabilities of the finite element method [75] and the Godunov method [2], turned out to be a promising method that allows building computational algorithms of an increased order of accuracy. For the numerical solution of dynamic problems of gas dynamics in [76], a very effective “Cabaret” scheme (the jump transfer method) was developed, which made it possible to advance in the numerical solution of problems of plasma dynamics. A review of finite volume methods (FVO) for solving systems of hyperbolic equations, which have gained considerable popularity in the last decade, is given in the monograph [77]. Numerical methods developed for solving problems of continuum mechanics have been successfully used in various applications. Thus, among the works devoted to the calculation of aerohydrodynamic properties of aircraft, the following are noted [2, 7, 8, 9, 17, 29 et al.], monographs devoted to the study of hypersonic flow around blunted bodies [78–80]. In [81], the problems of hypersonic flow around the deformable shell of an aircraft descending in dense layers of the atmosphere were considered, in [82] — the problem of supersonic flow around a system of bodies. The calculation of flows of an incompressible fluid stratified by density in the shallow water approximation is devoted to the work [83].

The problems of solar wind flow around the Earth›s magnetosphere using the equations of magnetohasodynamics were investigated in [84]; acoustic-gravitational waves arising in the atmosphere — in [85]. The motion of an asteroid in the Earth’s atmosphere, its interaction with the Earth’s surface, and the subsequent propagation of seismic waves in the Earth’s crust were considered in [86].

Examples of numerical solutions to problems of deformable solid mechanics can be found in [59, 35, 67−69, 74, etc.]. Wave processes and fracture processes in complex composite structures were studied in [87-89]; problems of interaction of concentrated energy flows and deformable targets — in [90−92]; seismic exploration — in [93−96]; Arctic shelf — in [97−99]; railway safety — in [100]; global intraplanetary seismics — in [101]; electromagnetic wave propagation — in [102−103]; medicine — [104−107]; the intensity of street traffic in megacities (graph problems) — in [108]; large power grids [109]; information flows in computer networks [110].

Of course, these articles cannot be called a review of applied works on numerical modeling of physical processes, because there are too many of them. However, they can give a definite picture of research in the field under consideration.

Systems of partial differential equations of the first order for two independent variables *t*, *x* have the form:

$$F_i\left(t, x, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}\right) = 0,\ i = 1 \div N. \tag{1}$$

Vector functions, $\overline{u} = \{u_1, ..., u_N\}^T$, having the first continuous derivatives and satisfying the system of equations (1), are the solution of this system.

System (2) resolved with respect to the derivative of one of the independent variables (*t* or *x*),

$$\frac{\partial u_i}{\partial t} = G_i\left(t, x, u, \frac{\partial u}{\partial x}\right) \tag{2}$$

is called the normal form.

If in the system of differential equations (1) all functions F_i ($i = 1 \div N$) are linear with respect to each of the quantities $u, \frac{\partial u}{\partial t}, \frac{\partial u}{\partial x}$, then such a system is linear with respect to the specified quantities.

If the system of partial differential equations of the first order (1) is quasi-linear, that it admits writing in the form:

$$\sum_{k=1}^{N} a_{ik} \frac{\partial u_k}{\partial t} + \sum_{k=1}^{N} b_{ik} \frac{\partial u_k}{\partial x} = f_i, i = 1 \div N, \tag{3}$$

where a_{ik}, b_{ik} depend on the independent variables *t*, *x* and solutions $\overline{u}$.

If they do not depend on *u*, the system is called semi-linear. If f_i does not depend on the solution of the system, then it is linear.

It is possible to represent system (3) in matrix form

$$D \frac{\partial u}{\partial t} + P \frac{\partial u}{\partial x} = f, \tag{4}$$

which, assuming that the matrix *D* is nonsingular, is represented as:

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = f. \tag{5}$$

For the case, respectively, of three or four independent variables, (5) has the form:

$$\frac{\partial u}{\partial t} + A_1 \frac{\partial u_1}{\partial x_1} + A_2 \frac{\partial u_2}{\partial x_2} = f, \tag{6}$$

$$\frac{\partial u}{\partial t} + A_1 \frac{\partial u_1}{\partial x_1} + A_2 \frac{\partial u_2}{\partial x_2} + A_3 \frac{\partial u_3}{\partial x_3} = f,$$

or

$$\frac{\partial u}{\partial t} + \sum_{k=1}^{K} A_k \frac{\partial u_k}{\partial x_k} = f, \text{ where } K = 2, \text{ or } 3. \tag{7}$$

In the future, we will consider a system of quasi-linear partial differential equations written in normal form for the one-dimensional case:

$$\frac{\partial u}{\partial t} + A \frac{\partial u}{\partial x} = f. \tag{8}$$

Assuming that all eigenvalues of matrix *A* are real and there is a basis of eigenvectors $\{\omega_i\}$ multiply (8) by the left eigenvector and

$$\omega_i \frac{\partial u}{\partial t} + \omega_i A \frac{\partial u}{\partial x} = \omega_i f, \tag{9}$$

or

$$\omega_i \left(\frac{\partial u}{\partial t} + \lambda_i \frac{\partial u}{\partial x}\right) = \omega_i f. \tag{10}$$

When disclosing scalar products in (10), there is:

$$\sum_{i=1}^{N}\omega_i\left(\frac{\partial u_i}{\partial t}+\lambda_i\frac{\partial u_i}{\partial x}\right)=\sum\omega_i f_i. \quad (11)$$

The expression in parentheses can be written as:

$$\frac{\partial u_i}{\partial t}+\omega_i\frac{\partial u_i}{\partial x}=\left(\frac{\partial u_i}{\partial t}\right)\Bigg|_{\frac{\partial x}{\partial t}=\lambda_i}, \quad (12)$$

where $\frac{\partial u_i}{\partial t}$ is the derivative of the desired function $u_i(t, x)$ in the direction $\frac{dx}{dt}=\lambda_i$.

Thus, a linear combination of derivatives $\left(\frac{du_i}{dt}\right)$ is obtained in (10).

The direction determined by an ordinary differential equation of the form:

$$\frac{dx}{dt}=\lambda_i \quad (13)$$

is called characteristic.

In the future, the system of partial differential equations (8) will be called hyperbolic (or a system of hyperbolic type equations) in some simply connected domain L, to which the quantities t, x, u belong, if two conditions are met at any point L:

– all corresponding values $\lambda_k(t, x, u)$ of the matrix A (t, x, u) are real;

– in a linear vector space R_N there exists an orthonormal basis $\{\omega_i\}_{i=1}^{i=N}$, composed of the left eigenvectors of the matrix A and satisfying the condition:

$$\det\Omega=\det\left\{\omega_i^k\right\}=\det\begin{Bmatrix}\omega_1^1\omega_2^1 & \ldots & \omega_N^1\\ - & - & -\\ \omega_1^N\omega_2^N & \ldots & \omega_N^N\end{Bmatrix}=0. \quad (14)$$

Sometimes a third condition is added to the above conditions — for the smoothness of the eigenvalues of the vectors of the matrix A (for example, in Petrovsky's definition, the condition is given that λ_k and $\left\{\omega_i^k\right\}$ must have the same smoothness as the elements of the matrix A (t, x, u).

The system of partial differential equations (8) under consideration is called hyperbolic in the narrow sense if at any point L_N the eigenvalues of the matrix are real and different. The definition of hyperbolicity of the system implies the equivalence of two systems of equations:

$$\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial x}=f,$$

$$\omega^k\left(\frac{\partial u}{\partial t}+\lambda_k\frac{\partial u}{\partial x}\right)=\omega_k f_k.$$

The system (11) is called the characteristic form of the original system of equations (8).

The characteristic form of the system under consideration can also be represented as:

$$\omega^k\left(\frac{\partial u}{\partial t}\right)=f_k, \quad (15)$$

where $\left(\frac{\partial u}{\partial t}\right)_k=\frac{\partial u}{\partial t}+\lambda_k\frac{\partial u}{\partial x}$.

If the eigenvalues and eigenvectors of the matrix A are in the system of equations

$$\frac{\partial u}{\partial t}+A\frac{\partial u}{\partial x}=0 \quad (16)$$

are constant, then the matrix A is represented as a product:

$$A=\Omega^{-1}\Lambda\Omega, \quad (17)$$

where Λ is the diagonal matrix consisting of the eigenvalues $\{\lambda_1,...,\lambda_N\}$ of matrix A, Ω — is the matrix whose rows are the left eigenvectors of A.

When multiplying (16) by Ω and entering Riemann variables $v = \Omega u$ a new system of the form is output:

$$\frac{\partial v}{\partial t} + \Lambda \frac{\partial v}{\partial x} = 0, \tag{18}$$

where $\Lambda =$ diag $\{\lambda_1, ..., \lambda_N\}$, or in scalar form $\frac{\partial v_k}{\partial t} + \lambda_k \frac{\partial v_k}{\partial x} = 0, \ \ k = 1 \div N.$

It can be seen that the initial system decays into N separate scalar transport equations, the solutions of which will be traveling waves

$$v_k = v_k\left(x - \lambda_k t\right), \ k = 1 \div N, \tag{19}$$

each of which propagates at a speed of λ_k, while maintaining its initial shape.

The general solution of the system is a superposition of traveling waves propagating at the specified speeds:

$$\bar{v} = \sum_{i=1}^{N} \omega^k v_k\left(x - \lambda_k t\right). \tag{20}$$

Riemann invariants

If the system of eigenvectors is orthonormal, then the values can be interpreted as the amplitudes of traveling waves. The functions are called Riemann invariants, and the system with (18) is a system in invariants.

Next, the concept of Riemann invariants is considered on a simple example — an acoustic system of their two scalar partial differential equations describing the propagation of plane sound waves:

$$\begin{cases} \frac{\partial u}{\partial t} + \frac{1}{\rho_0} \cdot \frac{\partial p}{\partial x} = 0 & (21.1) \\ \frac{\partial u}{\partial t} + \rho_0 c_0^2 \cdot \frac{\partial u}{\partial x} = 0, & (21.2) \end{cases}$$

where u is the velocity of the continuous medium, p is the pressure in the medium; ρ_0 is the density; c_0 is the speed of sound propagation in the medium.

If both of these equations are integrated over an arbitrary domain with a boundary G in the plane $\{t, x\}$ and go to contour integrals, this will lead to integral equations:

$$\begin{cases} \oint_G \rho_0 u dx - p dt = 0, \\ \oint_G \frac{\rho_2}{c_0} u dx - \rho_0 u dt = 0, \end{cases} \tag{22}$$

representing the laws of conservation of momentum and mass. In this case, the equation of state has the form: $p = c^2{}_0(\rho - \rho_0)$.

When multiplying the first equation (21.1) by $\rho_0 u$, and (21.2) by $p/(\rho_0 c^2{}_0)$ and adding them, the identity is derived:

$$\frac{\partial}{\partial t}\left(\rho_0 \frac{u^2}{2} + \frac{p^2}{2\rho_0 c_0^2}\right) + \frac{\partial}{\partial x}\left(pu\right) = 0, \tag{23}$$

from which it follows that for any closed circuit the law of conservation of energy of acoustic waves is valid:

$$\oint_G \left(\rho_0 \frac{u^2}{2} + \frac{p^2}{2\rho_0 c_0^2}\right) dx - pudt = 0. \tag{24}$$

Now it is necessary to bring the system (21) to the kinetic form. To do this, the second equation is multiplied by $(\rho_0 c_0)^{-1}$, then added to the first and subtracted from it, after which it turns out:

$$\begin{cases} \frac{\partial}{\partial t}\left(u + \frac{p}{\rho_0 c_0}\right) + c_0 \frac{\partial}{\partial x}\left(u + \frac{p}{\rho_0 c_0}\right) = 0, \\ \frac{\partial}{\partial t}\left(u - \frac{p}{\rho_0 c_0}\right) - c_0 \frac{\partial}{\partial x}\left(u - \frac{p}{\rho_0 c_0}\right) = 0. \end{cases} \tag{25}$$

Notation is introduced $R^+ = u + \frac{p}{\rho_0 c_0}, \ \ R^- = u - \frac{p}{\rho_0 c_0},$ (26)

the equations in Riemann invariants ($R^{\pm}$— Riemann invariant) are obtained:

$$\frac{\partial R^+}{\partial t}+c_0\frac{\partial R^+}{\partial x}=0,\ \frac{\partial R^-}{\partial t}-c_0\frac{\partial R^-}{\partial x}=0, \tag{27}$$

allowing you to write out their general solution:

$$R^+=f\left(x-c_0t\right),\ R^-=g\left(x+c_0t\right), \tag{28}$$

where *f*, *g* are functions defined from the conditions of the problem. Knowing the Riemann invariants, the values of the desired functions are obtained from (26):

$$\begin{cases} u=\frac{1}{2}\left[f\left(x-c_0t\right)+g\left(x+c_0t\right)\right], \\ p=\frac{\rho_0c_0}{2}\left[f\left(x-c_0t\right)-g\left(x+c_0t\right)\right]. \end{cases}$$

From the relations (27) it can be seen that the values R^+, R^- remain constant along straight lines $x_0-c_0t=\text{const}$ and $x_0+c_0t=\text{const}$, accordingly, their graphs move over time to the right (left) with speed c_0.

The straight lines

$$\frac{\partial x}{\partial t}=\pm c_0 \quad \text{or} \qquad x\pm c_0t=\text{const} \tag{29}$$

are called the characteristics of the system (21), which also needs to add initial conditions:

$$u\left(x,0\right)=u_0\left(x\right),\ p\left(x,0\right)=p_0\left(x\right), \tag{30}$$

from where follows:

$$u\left(x\right)=\frac{1}{2}\left[f\left(x\right)+g\left(x\right)\right],$$

$$p_0\left(x\right)=\frac{\rho_0c_0}{2}\left[f\left(x\right)-g\left(x\right)\right],$$

or

$$f\left(x\right)=u_0\left(x\right)+\frac{\rho_0(x)}{\rho_0c_0},$$

$$g\left(x\right)=u_0\left(x\right)+\frac{\rho_0(x)}{\rho_0c_0}.$$

In this case, the solution of the system (21) with the initial data (30) is represented as:

$$\begin{cases} u\left(t,x\right)=\frac{1}{2}\left[u_0\left(x-c_0t\right)+u_0\left(x+c_0t\right)\right]+\frac{1}{2\,\rho_0c_0}\left[p_0\left(x-c_0t\right)-p_0\left(x+c_0t\right)\right], \\ p\left(t,x\right)=\frac{1}{2}\left[p_0\left(x-c_0t\right)+p_0\left(x+c_0t\right)\right]+\frac{\rho_0c_0}{2}\left[u_0\left(x-c_0t\right)-u_0\left(x+c_0t\right)\right]. \end{cases} \tag{31}$$

Let, for example, the initial conditions have the following form:

$$\begin{gathered} u_0\left(x\right)=u_1,\ p_0\left(x\right)=p_1,\ x<X, \\ u_0\left(x\right)=u_2,\ p_0\left(x\right)=p_2,\ x>X, \end{gathered} \tag{32}$$

where u_i, $p_i(i=1,2)$ are constants, moreover, one of the equalities is fulfilled $u_1\neq u_2$ or $p_1\neq p_2$, or both at the same time.

The solution to this problem, which is not difficult to obtain, is given by the following relations:

$$\begin{gathered} u=u_1,\ p=p_1,\ x<X-c_0t, \\ u=u_2,\ p=p_2,\ x>X+c_0t, \\ u=\frac{u_1+u_2}{2}-\frac{p_1+p_2}{2\,\rho_0c_0}, \\ p=\frac{p_1+p_2}{2}-\rho_0c_0\frac{u_2+u_1}{2},\ X-ct<x<X+ct. \end{gathered} \tag{33}$$

The resulting solutions $u(t, x)$, $p(t, x)$, as can be seen, have discontinuities along the lines $x+c_0t=X$ and $x-c_0t=X$ and were formed from the initial discontinuity at the point x=X. For this reason, the considered problem is called the gap

decay problem. Generally speaking, these functions cannot formally be considered a solution to this problem due to the fact that they are not continuous. For this reason, they are called the generalized solution of the gap decay problem.
It is worth noting that the concept of invariants was introduced by Riemann in 1876.

Discussion and conclusions. More complex example is given — the solution of a one-dimensional system of gas dynamics equations:

$$\begin{cases} \frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+\frac{1}{\rho}\cdot\frac{\partial p}{\partial x}=0 & (34.1) \\ \frac{\partial p}{\partial t}+u\frac{\partial p}{\partial x}+c^2\rho\cdot\frac{\partial p}{\partial x}=0, & (34.2) \end{cases}$$

where u, ρ are the velocity and density of the gas; p is the pressure in the gas, c is the sound velocity of the gas, t, x are the time and coordinate.

The second equation (27) is multiplied by $(\rho c)^{-1}$ and added to the first (27). It turns out:

$$\left[\frac{\partial u}{\partial t}+(u+c)\frac{\partial u}{\partial x}\right]+\frac{1}{\rho c}\left[\frac{\partial p}{\partial t}+(u+c)\frac{\partial p}{\partial x}\right]=0 \tag{35}$$

or

$$\left(\frac{\partial u}{\partial t}\right)^{+}+\frac{1}{\rho c}\left(\frac{\partial u}{\partial t}\right)^{+}=0, \tag{36}$$

where $\left(\frac{\partial u}{\partial t}\right)^{+}$ is the derivative in the direction $\frac{\partial u}{\partial t}=(u+c)$.

Similarly, calculations are carried out with the replacement of c by $(-c)$ after which the quasi-linear system of equations is reduced to the characteristic form:

$$\begin{cases} \left(\frac{\partial S}{\partial t}\right)^{0}=0, \\ \left(\frac{\partial u}{\partial t}\right)^{+}+\frac{1}{\rho c}\left(\frac{\partial p}{\partial t}\right)^{+}=0, \\ \left(\frac{\partial u}{\partial t}\right)^{-}+\frac{1}{\rho c}\left(\frac{\partial p}{\partial t}\right)^{-}=0, \end{cases} \tag{37}$$

where $\left(\frac{\partial}{\partial t}\right)^{0}=\frac{\partial}{\partial t}+u\frac{\partial}{\partial x}$,

and the first equation (30) expresses the law of conservation of entropy along the trajectory of the particle, i. e. on the trajectory described by an ordinary equation of the form:

$$\frac{\partial X}{\partial t}=u(t,X),\ X(0)=X_0,$$

where the function $X(t)$ is the trajectory of the particle.

In the case of an isentropic flow, i. e. when

$$p=A\rho^{\gamma}(A=\text{const}), \tag{38}$$

and, accordingly,

$$c^2=A\gamma\rho^{\gamma-1},\left(c=\sqrt{\frac{\partial p}{\partial \rho}}\right),$$

the expression $\frac{\partial p}{\partial c}$ becomes differential:

$$\frac{1}{\rho c}dp=\frac{2}{\gamma-1}dc.$$

Then, after adding a multiplier $(\rho c)^{-1}$ under the sign of differentiation of the obtained equations, it turns out:

$$\begin{cases} \frac{\partial S}{\partial t}+u\frac{\partial S}{\partial t}=0 \\ \frac{\partial R^{+}}{\partial t}+(u+c)\frac{\partial R^{+}}{\partial x}=0 \\ \frac{\partial R^{-}}{\partial t}+(u-c)\frac{\partial R^{-}}{\partial x}=0, \end{cases} \tag{39}$$

or

$$\begin{cases} \left(\dfrac{\partial S}{\partial t}\right)^0 = 0 \\ \left(\dfrac{\partial R^+}{\partial t}\right) = 0 \\ \left(\dfrac{\partial R^-}{\partial t}\right) = 0, \end{cases} \tag{40}$$

where $R^+ = u + \dfrac{2c}{\gamma - 1}$, $R^- = u - \dfrac{2c}{\gamma - 1}$ (41)

Riemann invariants for a one-dimensional quasi-linear system of gas dynamics equations that retain their values on the trajectories of the equations,

$$\frac{\partial X^+}{\partial t} = u \pm c. \tag{42}$$

It is obvious that through the values of the Riemann invariants and entropy, which are found from the solutions of ordinary differential equations, the remaining functions (u, p, ρ), describing the gas flow are calculated. However u, c are the functions of S, $R^{\pm}$, themselves, so it is impossible to find a solution to these equations in quadratures, in any case. However, the exact solution is in the special case for $\gamma = 3$ (detonation products).

Since in this case $R^{\pm} = u \pm c$, the trajectories $X^{\pm}$ $(t, X^{\pm}_0)$ are families of straight lines with constant slope.

The characteristic form of the gas dynamics equations makes it possible to understand how to set the boundary conditions correctly. For example, the left boundary of the integration domain is considered. Three characteristics with slopes u, $(u + c)$, $(u - c)$ pass through any point of it. Those of them whose slopes are positive are called entering the integration domain. Thus, it is necessary to set as many conditions on the left border as there are characteristics included in the area; similarly, on the right border.

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