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A family of inverse characteristics methods

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Abstract

Introduction. The main idea of the grid-characteristic method is to take into account the characteristic properties, systems of hyperbolic equations, and the finite velocity of propagation of perturbations in the simulated media.

Materials and methods. The simplest hyperbolic equation is a one-dimensional linear transfer equation. To increase the order of approximation of the grid-characteristic scheme to the second, you can use the Bim-Warming scheme. If we use a four-point pattern, we get a central Lax-Vendroff scheme. Difference schemes for the linear transfer equation can be obtained using the method of indefinite coefficients.

Results. The grid-characteristic scheme admits a conservative variant, which is relevant if there are discontinuities (shock waves, shock waves) inside the integration domain, while the original system of equations for a matrix with constant coefficients, in partial derivatives, should be presented in a divergent form.

Discussion and conclusions. The construction is performed similarly, when numerically solving a three-dimensional problem, in the case of upper and lower bounds, after scalar multiplication of the scheme by eigenvectors, relations approximating the compatibility conditions with the first order of accuracy are obtained.

Keywords: grid-characteristic method, hyperbolic type equations, transfer equation, Beam-Warming scheme, Lax-Wendroff scheme, method of indefinite coefficients, compatibility conditions.

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Научная статья

Семейство методов обратных характеристик

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Аннотация

Введение. Основной идеей сеточно-характеристического метода является учет характеристических свойств, систем уравнений гиперболического типа, конечной скорости распространения возмущений в моделируемой средах.

Материалы и методы. Простейшим уравнением гиперболического типа является одномерное линейное уравнение переноса. Для повышения порядка аппроксимации сеточно-характеристической схемы до второго, можно использовать схему Бима-Уорминга. Если использовать четырехточечный шаблон, то получим центральную схему Лакса-Вендроффа. Разностные схемы для линейного уравнения переноса можно получать, используя метод неопределенных коэффициентов.

Результаты исследования. Сеточно-характеристическая схема допускает консервативный вариант, актуальный, если внутри области интегрирования имеются разрывы (скачки уплотнения, ударные волны) при этом исходная система уравнений для матрицы с постоянными коэффициентами, в частных производных должна быть представлена в дивергентной форме.

Обсуждение и заключения. При численном решении трехмерной задачи построение производится аналогично, в случае верхней и нижней границ, после скалярного умножения схемы на собственные векторы получены соотношения аппроксимирующие с первым порядком точности условия совместимости.

Ключевые слова: сеточно-характеристический метод, уравнения гиперболического типа, уравнение переноса, схема Бима-Уорминга, схема Лакса-Вендроффа, метод неопределенных коэффициентов, условия совместимости.

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Introduction. The main idea of the grid-characteristic method is to take into account the characteristic properties, systems of hyperbolic equations, and the finite velocity of propagation of perturbations in the simulated media. Reviews of relevant works are given in monographs [1–4].

The simplest hyperbolic equation is a one-dimensional linear transfer equation:

$$\frac{\partial u}{\partial t}+a\frac{\partial u}{\partial x}=0,\ a=\text{const},\ a>0, \tag{1}$$

and the simplest difference scheme that takes into account the characteristic properties of (1) is the “corner” scheme, or the Courant-Izakson-Riess scheme:

$$u_m^{n+1}=\left(1-\sigma\right)u_m^n+\sigma u_{m-1}^n,\ \sigma=\frac{a\tau}{h}. \tag{2}$$

This scheme takes into account the direction of the characteristic of this equation and can be obtained by linear interpolation of the numerical solution from its known values at the nodes x_m, x_{m-1} calculation grid. This scheme can be represented in the following forms:

$$u_m^{n+1}=u_m^n-\sigma\begin{cases}u_{m+1}^n-u_m^n,\ a<0,\\ u_m^n-u_{m-1}^n,\ a<0,\end{cases}$$

(the difference is selected taking into account the slope of the characteristic):

$$u_m^{n+1}=u_m^n-\frac{\tau}{h}\left[a^+\left(u_{m+1}^n-u_m^n\right)+a^-\left(u_m^n-u_{m-1}^n\right)\right],$$

где $a^+=0{,}5\left(a+|a|\right)$, $a^-=0{,}5\left(a-|a|\right)$; $u_m^{n+1}=u_m^n-\frac{\sigma}{2}\left(u_{m+1}^n-u_{m-1}^n\right)+\frac{|\sigma|}{2}\left(u_{m+1}^n-2u_m^n+u_{m-1}^n\right)$

(a scheme with an explicit allocation of a dissipative term that ensures its stability):

$$u_m^{n+1}=u_m^n-\sigma\left(\Phi_{m+1/2}^n-\Phi_{m-1/2}^n\right),$$

where

$$\Phi_{m+1/2}^n=\frac{1}{2}\left[a\left(u_{m+1}^n-u_m^n\right)-|a|\left(u_{m+1}^n-u_m^n\right)\right]$$

(streaming form).

$$\Phi_{m-1/2}^n=\frac{1}{2}\left[a\left(u_m^n-u_{m-1}^n\right)-|a|\left(u_m^n-u_{m-1}^n\right)\right]$$

In the case of a nonlinear transfer equation:

$$\frac{\partial u}{\partial t}+\frac{\partial f}{\partial x}=0,\ f=\frac{u^2}{2}, \tag{3}$$

the “corner” scheme, taking into account the directions of characteristics, can be presented in the following form:

$$u_m^{n+1}=u_m^n-\frac{\tau}{h}\begin{cases}f_{m+1}^n-f_m^n,\ u_m^n<0\\ f_m^n-f_{m-1}^n,\ u_m^n>0.\end{cases}$$

The reduced difference scheme has the first order of approximation in time and coordinate. To increase the order of approximation of the grid-characteristic scheme to the second, you can use the Bim-Warming scheme, which can be obtained on a 4-point template $\{x_{m-2}^n,x_{m-1}^n,x_m^n\}$ by quadratic interpolation of the solution on the n-th time layer by nodes $x_{m-2}^n,x_{m-1}^n,x_m^n$:

$$u_m^{n+1} = u_m^n - \sigma\left(u_m^n - u_{m-1}^n\right) - \frac{\sigma}{2}\left(1-\sigma\right)\left(u_m^n - 2u_{m-1}^n + u_{m-2}^n\right). \tag{4}$$

If use four-point template $\left\{x_m^{n+1}, x_{m-1}^n, x_m^n, x_{m+1}^n\right\}$,can get a central Lax-Vendroff scheme:

$$u_m^{n+1} = u_m^n - \frac{\sigma}{2}\left(u_{m+1}^n - u_{m-1}^n\right) + \frac{\sigma^2}{2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right). \tag{5}$$

This scheme, like the Bim-Warming scheme, has a second-order approximation in time and space Ơ $(\tau^2 + h^2)$, which is verified by decomposing grid functions $u_{m\pm1}^n$ into a Taylor series, and is stable when the Courant condition $\tau \le h/a$, is met, which can be obtained using the Neumann spectral stability feature. When adding a point x_{m-2}; to the template corresponding to the Lax-Wendroff scheme, we obtain a scheme of the 3rd order of approximation on a four-point template $\left\{x_m^{n+1}, x_{m-1}^n, x_m^n, x_{m+1}^n\right\}$:

$$\begin{gathered} u_m^{n+1} = u_m^n - \frac{\sigma}{6}\left(2u_{m+1}^n + 3u_m^n - 6u_{m-1}^n + u_{m-2}^n\right) + \frac{\sigma^2}{2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right) - \\ -\frac{\sigma^3}{6}\left(u_{m+1}^n + 3u_{m-1}^n - 3u_m^n - u_{m-2}^n\right). \end{gathered} \tag{6}$$

To increase the order of accuracy of the scheme, we add a point x_{m+2} to the last template, the difference scheme of the 4th order of approximation will have the form:

$$\begin{gathered} u_m^{n+1} = u_m^n - \frac{\sigma}{12}\left(8u_{m+1}^n - 8u_{m-1}^n - u_{m+2}^n + u_{m-2}^n\right) + \frac{\sigma^2}{24}\left(16u_{m+1}^n - 30u_m^n + 16u_{m-1}^n - u_{m+2}^n - u_{m-2}^n\right) - \\ -\frac{\sigma^3}{12}\left(2u_{m-1}^n - 2u_{m+1}^n + u_{m+2}^n - u_{m-2}n\right) + \frac{\sigma^4}{24}\left(6u_m^n - 4u_{m+1}^n - 4u_{m-1}^n + u_{m+2}^n + u_{m-2}^n\right). \end{gathered} \tag{7}$$

If a linear acoustic system of equations of the form is considered

$$\begin{cases} \dfrac{\partial u}{\partial t} + \dfrac{1}{\rho} \cdot \dfrac{\partial p}{\partial x} = 0 \\ \dfrac{\partial p}{\partial t} + \rho c^2 \cdot \dfrac{\partial u}{\partial x} = 0, \end{cases} \tag{8}$$

where u, p is the local velocity of the medium and pressure, ρ is the density, c is the speed of sound, then, as noted in the second chapter, it can be reduced to the form:

$$\begin{cases} \dfrac{\partial r}{\partial t} + a\dfrac{\partial r}{\partial x} = 0 \\ \dfrac{\partial s}{\partial t} - a\dfrac{\partial s}{\partial x} = 0, \end{cases} \tag{9}$$

where $r = u + \dfrac{p}{\rho a}$, $s = u - \dfrac{p}{\rho a}$ — are Riemann invariants.

In this case, the grid-characteristic method can be presented in the following form:

$$\begin{cases} r_m^{n+1} = \left(1-\sigma\right) \cdot r_m^n + \sigma r_{m-1}^n \\ s_m^{n+1} = \left(1-\sigma\right) \cdot s_m^n + \sigma s_{m+1}^n. \end{cases} \tag{10}$$

The corresponding template has the form.

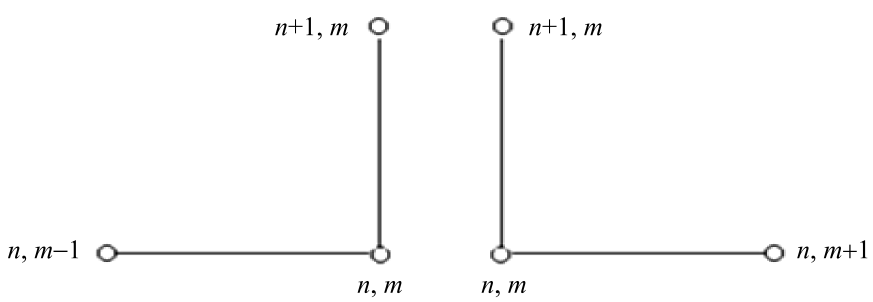

Fig. 1. Template of the grid-characteristic method for a linear acoustic system of equations

The given difference schemes can be obtained as follows.

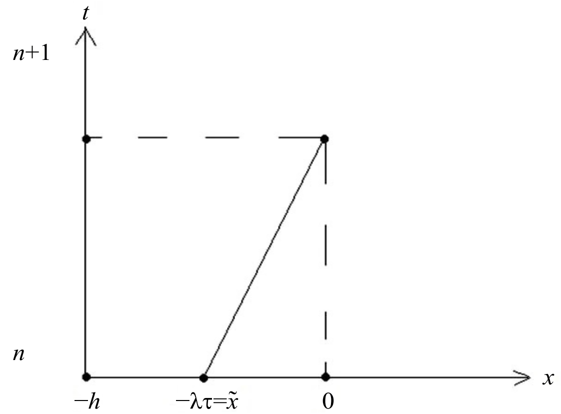

Fig. 2. Two-point template ($x_n = 0, x_m = –h$)

Figure 2 shows a two-point template ($x_n = 0, x_m = –h$), on which, in the case of a 1st-order scheme, a 1st-order polynomial is constructed $P_1(x)=a_1x + a_0$, whose coefficients are found from the conditions:

$$u_m^n = a_1 x_m + a_0,\ u_{m-1}^n = a_1 x_{m-1} + a_0 :$$

$$a_1 = \left(u_m^n + u_{m-1}^n\right)/h,\quad a_0 = u_m^n.$$

Then get:

$$P\left(x\right) = \frac{u_m^n - u_{m-1}^n}{h} x + u_m^n.$$

Since $u_m^{n+1} = u\left(\tilde{x}\right) = u\left(-\lambda\tau\right)$, get the "left corner" scheme:

$$u_m^{n+1} = u_m^n - \sigma\left(u_m^n - u_{m-1}^n\right).$$

If add a point x_{m+1}^n , to this template, get a second-order polynomial: $P_2\left(x\right) = a_2 x^2 + a_1 x + a_0$, whose coefficients are in the same way (from the conditions for passing the polynomial $P(x)$ through the values of the function at the nodes of the template):

$$a_2 = \left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right)/2h^2,$$

$$a_1 = \left(u_{m+1}^n - u_{m-1}^n\right)/2h,$$

$$a_0 = u_m^n.$$

In this case, we get the Lax-Wendroff scheme. When adding a point x_{m-2} to this template, we get a third-order polynomial: $P_3\left(x\right) = a_3 x^3 + a_2 x^2 + a_1 x + a_0$
with coefficients that are from the same conditions:

$$a_3 = \left(u_{m+1}^n + 3u_m^n - 3u_{m-1}^n - u_{m-2}^n\right)/6h^3,$$

$$a_2 = \left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right)/2h^2,$$

$$a_1 = \left(2u_{m+1}^n + 3u_m^n - 6u_{m-1}^n + u_{m-2}^n\right)/6h,$$

$$a_0 = u_m^n.$$

he Rusanov scheme of the third order of accuracy is obtained. If we add a point x_{m+2}, to the template, we get a polynomial of the fourth degree: $P_u\left(x\right) = a_4 x^4 + a_3 x^3 + a_2 x^2 + a_1$
with coefficients:

$$a_4 = \left(6u_m^n - 4u_{m+1}^n - 4u_{m-1}^n + u_{m-2}^n\right)/24h^4,$$

$$a_3 = \left(2u_{m-1}^n - 2u_{m+1}^n + u_{m+2}^n + u_{m-2}^n\right)/12h^3,$$

$$a_2 = \left(16u_{m+1}^n - 30u_m^n + 16u_{m-1}^n - u_{m+2}^n - u_{m-2}^n\right)/24h^2,$$

$$a_1 = \left(8u_{m+1}^n - 8u_{m-1}^n - u_{m+2}^n + u_{m-2}^n\right)/12h,$$

$$a_0 = u_m^n,$$

and, accordingly, a scheme of the fourth order of approximation.

Difference schemes for the linear transfer equation can also be obtained using the method of indefinite coefficients. For example, on the template

$$\left\{\left(t^n, x_{m+1}\right), \left(t^n, x_m\right), \left(t^n, x_{m-1}\right), \left(t^n, x_{m-2}\right), \left(t^n, x_{m-2}\right)\right\}$$

can represent all linear schemes with undefined coefficients $\left\{\alpha_j^0; j=-2,-1,0,1\right\}$ in the following form:

$$u_m^{n+1} = \alpha_{-2}^0 u_{m-2}^n + \alpha_{-1}^0 u_{m-1}^n + \alpha_0^0 u_m^n + \alpha_1^0 u_{m+1}^n.$$

After decomposing the grid functions into a Taylor series:

$$u_m^{n+1} = u_m^n + \tau\left(u_t'\right)_m^n + \frac{\tau^2}{2}\left(u_t''\right)_m^n + \frac{\tau^3}{6}\left(u_t'''\right)_m^n + O'(\tau)^4,$$

$$u_{m+1}^n = u_m^n + h\left(u_x'\right)_m^n + \frac{h^2}{2}\left(u_x''\right)_m^n + \frac{h^3}{6}\left(u_t'''\right)_m^n + O'\left(h^4\right),$$

$$u_{m+1}^n = u_m^n + h\left(u_x'\right)_m^n + \frac{h^2}{2}\left(u_x''\right)_m^n + \frac{h^3}{6}\left(u_t'''\right)_m^n + O'\left(h^4\right),$$

$$u_{m-2}^n = u_m^n + 2h\left(u_x'\right)_m^n + 2h\left(u_{xx}''\right)_m^n - \frac{4h^3}{3}\left(u_x'''\right)_m^n + O'\left(h^4\right),$$

taking into account the consequences of the linear transfer equation:

$$u_t' = -\lambda u_x', \quad u_{tt}'' = \lambda^2 u_{xx}'', \quad u_t''' = -\lambda^3 u_x'',$$

$$u_{tx}'' = -\lambda u_{xx}, \quad u_{ttx}'' = \lambda^2 u_{xx}''', \quad u_{txx}''' = -\lambda u_x'''$$

and by equating the coefficients in the left and right sides of equation (10), we obtain the approximation conditions:

$$\begin{cases} \lambda_{-2}^0 + \lambda_{-1}^0 + \lambda_0^0 + \lambda_1^0 = 1, \\ 2\lambda_{-2}^0 + \lambda_{-1}^0 - \lambda_1^0 = \sigma, \end{cases} \tag{11}$$

for a first-order scheme:

$$\begin{cases} \lambda_{-2}^0 + \lambda_{-1}^0 + \lambda_0^0 + \lambda_1^0 = 1, \\ 2\lambda_{-2}^0 + \lambda_{-1}^0 - \lambda_1^0 = \sigma, \\ 4\lambda_{-2}^0 + \lambda_{-1}^0 + \lambda_1^0 = \sigma^2, \end{cases} \tag{12}$$

for a second-order scheme:

$$\begin{cases} \lambda_{-2}^0 + \lambda_{-1}^0 + \lambda_0^0 + \lambda_1^0 = 1, \\ 2\lambda_{-2}^0 + \lambda_{-1}^0 - \lambda_1^0 = \sigma, \\ 4\lambda_{-2}^0 + \lambda_{-1}^0 + \lambda_1^0 = \sigma^2, \\ 8\lambda_{-2}^0 + \lambda_{-1}^0 - \lambda_1^0 = \sigma^3, \end{cases} \tag{13}$$

for a fourth-order precision scheme.

If we find all the undefined coefficients from (13) and substitute them into (10), then we can find the first differential approximation of the scheme:

$$u_t' + \lambda u_x' = \frac{u_x^{IV}}{24}\lambda\left(\lambda^4\tau^3 + 2\lambda^2\tau h + \lambda^2 h^2 - 2h^3\right), \tag{14}$$

that is, on the considered template, it is possible to build difference schemes no higher than the third order of approximation: Ơ $(\tau^3 + h^3)$.

In the case of a template of the general form of sets of difference schemes can be represented in the following form:

$$\begin{gathered} u_m^{n+1} = \sum \lambda_j^i(\sigma) u_{m+j}^{n+i}; \\ i = 1, 0, -1, \ldots; j = 0, \pm 1, \pm 2, \ldots \end{gathered} \tag{15}$$

The conditions of approximation of the first order in this case can be written as:

$$\sum_{i,j} \alpha_j^i = 1, \quad \sum_{i,j}\left(j - i\sigma\right) \cdot \alpha_j^i(\sigma) = -\sigma.$$

The conditions of approximation of the second and higher orders will have the form:

$$\sum (j - i\sigma)^p \cdot \alpha_j^i = (-\sigma)^p;\ p = 2,3,...$$

So, for a 4-point template and a two-layer explicit difference scheme:

$$\left\{t^{n+1}, x_m; t^n, x_{m-2}; t^n; x_{m-1}; t^n; x_m; t^n; x_{m+1}\right\}$$

can represent the entire family of linear difference schemes with undefined coefficients

$$L_j^0, (j = -2, -1, 0, 1)$$

in the following form:

$$u_m^{n+1} = L_{-2}^0 \cdot u_{m-2}^n + L_{-1}^0 \cdot u_{m-1}^n + L_0^0 \cdot u_m^n + L_1^0 \cdot u_{m+1}^n. \tag{16}$$

The approximation conditions of the first, second and third order on the solutions of the problem are linear with respect to indefinite coefficients of equality:

$$\begin{cases} \alpha_{-2}^0 + \alpha_{-1}^0 + \alpha_0^0 + \alpha_1^0 = 1, \\ 2\alpha_{-2}^0 + \alpha_{-1}^0 - \alpha_1^0 = \sigma, \\ 4\alpha_{-2}^0 + \alpha_{-1}^0 + \alpha_1^0 = \sigma^2, \\ 8\alpha_{-2}^0 + \alpha_{-1}^0 - \alpha_1^0 = \sigma^3. \end{cases}$$

These conditions can be easily obtained if we decompose all the projection values of the exact solution of the problem when substituting into the Taylor series relative to any point of our grid pattern, for example, (t^n, x_m):

$$\begin{cases} u_m^{n+1} = u_m^n + \tau (u_t)_m^n + \dfrac{\tau^2}{2}(u_{tt})_m^n + \dfrac{\tau^3}{6}(u_{ttt})_m^n + O(\tau^4), \\ \alpha_1^0 u_{m+1}^n = \alpha_1^0 \left(u_m^n + h(u_x)_m^n + \dfrac{h^2}{2}(u_x)_m^n + \dfrac{h^3}{6}(u_{xxx})_m^n + O(h^4) \right), \\ \alpha_0^0 u_m^n = \alpha_0^0 u_m^n, \\ \alpha_{-1}^0 u_{m-1}^n = \alpha_{-1}^0 \left(u_m^n - h(u_x)_m^n + \dfrac{h^2}{2}(u_x)_m^n + \dfrac{h^3}{6}(u_{xxx})_m^n + O(h^4) \right). \\ \alpha_{-2}^0 u_{m-2}^n = \alpha_{-2}^0 \left(u_m^n - 2h(u_x)_m^n + 2h^2(u_x)_m^n - \dfrac{4h^3}{3}(u_{xxx})_m^n + O(h^4) \right) \end{cases}$$

The consequences of the equation should be used:

$$u_t = -\lambda u_t, u_{tt} = \lambda^2 u_{xx}, u_{tx} = -2\lambda u_{xx},$$

$$u_{ttt} = -\lambda^3 u_{xxx}, u_{ttx} = \lambda^2 u_{xxx}, u_{txx} = -\lambda u_{xxx}.$$

Using these expressions, you can get rid of partial derivatives in time by expressing them in terms of spatial derivatives:

$$u_m^{n+1} = u_m^n - \lambda\tau (u_x)_m^n + \frac{\lambda^2\tau^2}{2}(u_{xx})_m^n - \frac{\lambda^3\tau^3}{6}(u_{xxx})_m^n + O(\tau^4).$$

By equating the values u^n_m, $(u_x)^n_m$, $(u_{xx})^n_m$, $(u_{xxx})^n_m$ in the left and right parts, we obtain the approximation conditions.

In this case, $\alpha_{-2}^i = 0$ we get a difference Lax-Wendroff scheme a$\alpha_{-2}^i = \frac{\sigma}{2}(\sigma - 1)$, Bim-Warming scheme at $\alpha_{-2}^0 = \frac{\sigma}{6}(\sigma^2 - 1)$ and a third-order approximation scheme (Rusanov), the only one on the template under consideration.

Research results. Consider the case of a linear system of transfer equations of the form:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0, \tag{17}$$

where $A(n \times m)$ is a matrix with constant elements and real eigenvalues λ, $i = 1,..., n$. Then this matrix is represented as:

$$A = \Omega^{-1}\Lambda\Omega,$$

where Λ is a diagonal matrix consisting of eigenvalues: $\Lambda = \text{diag}\ \{\lambda_1\lambda_2, ..., \lambda_n\} = \text{diag}\{\lambda_i\}$; $i = 1$, which are determined from the equation: $\det(A - \lambda E) = 0$,

where E is a unit matrix; Ω is a matrix, whose rows are the left eigenvectors of the matrix A p to their length from a system of linear homogeneous equations

$$\omega_i\left(A-\lambda_i E\right)=0;\ i=\ 1,...,n.$$

In this case (17) we write in the form:

$$\frac{\partial u}{\partial t}+\Omega^{-1}\Lambda\Omega\frac{\partial u}{\partial x}=0,$$

then we multiply this system by Ω:

$$\Omega\frac{\partial u}{\partial t}+\Lambda\Omega\frac{\partial u}{\partial x}=0. \tag{18}$$

By introducing the Riemann variables $w = \Omega u$ present the last equation as:

$$\frac{\partial w}{\partial t}+\Lambda\frac{\partial w}{\partial x}=0, \tag{19}$$

from where it can be seen that the system decays into separate equations, the exact solutions of which are traveling waves:

$$w_i = w_i\left(x-\lambda_i t\right);\ i=1,..,n.$$

The functions w_i are called Riemann invariants of the system (17); λ_i are the propagation velocities of perturbations in the medium. The general solution of the system of partial differential equations under consideration is the sum of n traveling waves, each of which propagates with its own characteristic velocity λ_i:

$$U=\sum_{i=1}^{n} b_i^2 w_i\left(x-\lambda_i t\right).$$

Note that if the vectors b_i are normalized, i.e. $|\ b_i| = 1$, the values w_i can be interpreted as the amplitudes of the corresponding traveling waves.

Let's represent (18) in scalar form:

$$\omega_i\frac{\partial u_i}{\partial t}+\lambda_i\omega_i\frac{\partial u_i}{\partial x}=0,\ i=1,..,n. \tag{20}$$

Next, we introduce a difference grid in the integration domain $\{t\geq 0, 0\leq x\leq X\}$ the difference grid

$$\left\{t^n=n\tau; n=0,1,...; x_i=\left(m-1\right)h; m=1,..,M; X=\left(M-1\right)h\right\}$$

and carry out the approximation (20) using finite difference relations taking into account the slope of the characteristics:

$$\frac{\partial x}{\partial t}=\lambda_i$$

(or taking into account the sign λ_i):

$$\omega_i\frac{u_m^{n+1}-u_m^n}{\tau}\mp\lambda_i\omega_i\frac{u_{m\mp1}^n-u_m^n}{h}+\omega_i f=0,\ i=\ 1,...,n. \tag{21}$$

The latter relation can be represented in matrix form if put:

$$|\Lambda|=\{|\lambda_i|\},\ \Lambda^+=\frac{1}{2}\left(\Lambda+|\Lambda|\right),\ \Lambda^-=\frac{1}{2}\left(\Lambda-|\Lambda|\right): \tag{22}$$

$$\Omega\left(u_m^{n+1}-u_m^n\right)-\sigma\Lambda^+\Omega\left(u_{m-1}^n-u_m^n\right)+\sigma\Lambda^-\Omega\left(u_{m+1}^n-u_m^n\right)+\tau\Omega f=0.$$

Since the matrix Ω is not degenerate, the resulting difference scheme can be represented as follows:

$$u_m^{n+1}=u_m^n-\sigma A\left(u_{m+1/2}^n-u_{m-1/2}^n\right)-\tau f+\frac{\sigma}{2}\Omega^{-1}|\Lambda|\Omega\left(u_{m-1}^n-2u_m^n+u_{m+1}^n\right),$$
$$u_{m\pm1/2}^n=\frac{1}{2}\left(u_{m\pm1}^n+u_m^n\right). \tag{23}$$

In this scheme, the summand is clearly highlighted:

$$\frac{\sigma}{2}\Omega^{-1}|\Lambda|\Omega\left(u_{m-1}^n-2u_m^n+u_{m+1}^n\right),$$

which ensures its stability when the CFL condition (Courant — Friedrichs— Levy) is met:

$$\sigma\leq\frac{1}{\max|\lambda_i|}.$$

An additional "dissipative" term is introduced into the initial system, which ensures the monotony of the obtained scheme of the first order of approximation, but is also a non-physical source having a difference origin (approximation viscosity).

This difference scheme, when generalized to the multidimensional case, belongs to the class of Friedrichs positive-definite schemes, as can be seen from the following representation of it:

$$u_m^{n+1} = \left(E - \sigma\Omega^{-1}\left|\Lambda\right|\Omega\right)u_m^n + \sigma\Omega^{-1}\Lambda^{+}\Omega u_{m-1}^n - \sigma\Omega^{-1}\Lambda u_{m-1}^n - \tau f.$$

The numerical method under consideration can also be presented in the following form:

$$u_m^{n+1} = u_m^n - \sigma\left(\Omega^{-1}\Lambda\Omega\right)\Delta u + \sigma^{\beta}\left(\Omega^{-1}\left|\Lambda\right|^{\beta}\Omega\right)\Delta^2 u,$$

where

$$\Delta u = \frac{1}{2}\left(u_m^{n+1} - u_{m-1}^n\right)/2;\ \Delta^2 u = \frac{1}{2}\left(u_{m+1}^n - 2u_m^n + u_{m-1}^n\right).$$

This scheme has the first order of approximation at β = 1 and the second at β = 2.

Let's imagine this scheme as a scheme with a weighting factor α (0 ≤ α ≤ 1):

$$u_m^{n+1} = u_m^n - \sigma\left(\Omega^{-1}\Lambda\Omega\right)\Delta u + 2\sigma\left(\Omega^{-1}\left|\Lambda\right|^{\beta}\Omega\right)\Delta^2 u + (1-\alpha)\sigma^2\left(\Omega^{-1}\Lambda\Omega\right)\Delta^2 u.$$

At α = 1 get a scheme of the first order of approximation, at α = 0 — the second. At 0 < α < 1 obtain a scheme having a formal first order, but with proper choice of value α, it will improve the description of numerical solutions with large gradients by reducing the approximation of viscosity [7].

The grid-characteristic scheme also admits a conservative variant, relevant if there are discontinuities (shock waves, shock waves) inside the integration domain, while the original system of equations for a matrix with constant coefficients, in partial derivatives, should be presented in a divergent form:

$$\frac{\partial u}{\partial t} + \Lambda\frac{\partial \Phi}{\partial x} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\left(Au\right) = \frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0, \tag{24}$$

where $A = \frac{\partial \Phi}{\partial u}$; $\Phi = Au$; since the system in question is hyperbolic, then: $A = \Omega^{-1}\Lambda\Omega$.

In this case, the difference scheme, which has the property of conservativeness, i.e. ensuring the implementation of conservation laws, can be presented in the following form:

$$u_m^{n+1} = u_m^n - \sigma\left(\Phi_{m+1/2}^n - \Phi_{m-1/2}^n\right) + \sigma\left[D_{m+1/2}\left(u_{m+1}^n - u_m^n\right)/h - D_{m-1/2}\left(u_m^n - u_{m-2}^n\right)/h\right], \tag{25}$$

where $\Phi_{m\pm1/2}^n = \frac{1}{2}\left(\Phi_{m\pm1}^n + \Phi_m^n\right)$.

It is clear that in the resulting difference scheme, in addition to physical flows through the cell boundaries there are also flows of difference origin:

$$Du_{xx}'', \text{where } D = \frac{h}{2}\Omega^{-1}\left|\Lambda\right|\Omega.$$

However, for neighboring cells on their common border, they are equal and opposite in sign: therefore, in general, inside the integration area, the scheme is conservative, which is checked by their addition over the entire area.

Note that among the schemes of the first order of approximation, this scheme has a minimum approximation viscosity. In the case of three independent variables $\{t, x, y\}$ conservative and non-conservative schemes will have, respectively, the following form:

$$\begin{gathered} u_{ml}^{n+1} = u_{ml}^n - \frac{\tau}{h_1}\left(\Phi_{m\pm1/2,l}^n - \Phi_{m-1/2,l}^n\right) - \frac{\tau}{h_2}\left(G_{m,l+1/2}^n - G_{m,l-1/2}^n\right) - \frac{\tau}{h_1}\cdot \\ \cdot\left[D_{m+1/2,l}^1\left(u_{m+1,l}^n - u_{ml}^n\right)/h_1 - D_{m-1/2,l}^1\left(u_{ml}^n - u_{m-1,l}^n\right)/h_1\right] + \\ + \frac{\tau}{h_2}\left[D_{m,l+1/2}^2\left(u_{m,l+1}^n - u_{ml}^n\right)/h_2 - D_{m,l-1/2}^2\left(u_{m,l}^n - u_{m,l-1}^n\right)/h\right], \\ u_{m,l}^{n+1} = u_{m,l}^n + \frac{\tau}{h_1}\left[\Omega_1^{-1}\Lambda_1^{+}\Omega_1\left(u_{m-1,l}^n - u_{ml}^n\right) - \Omega_1^{-1}\Lambda_1^{-}\Omega_1\left(u_{m+1,l}^n - u_{ml}^n\right)\right] + \\ + \frac{\tau}{h_2}\left[\Omega_2^{-1}\Lambda_2^{+}\Omega_2\left(u_{m,l-1}^n - u_{ml}^n\right) - \Omega_2^{-1}\Lambda_2^{-}\Omega_2\left(u_{m,l+1}^n - u_{ml}^n\right)\right], \end{gathered} \tag{26}$$

where $D^k = \frac{h_k}{2}\Omega_k^{-1}|\Lambda_k|\Omega_k;\ k = 1,2;$

the matrices Λ_k, $\Lambda^{\pm}_k$, Ω_k, as in the one-dimensional case, consist of eigenvalues and eigenvectors of matrices A_k; in the integration domain $\{t \geq 0; x \leq x \leq X, 0 \leq y \leq Y\}$ difference grid is introduced:

$$\left\{t^n = n\tau,\ n = 0,1,...;\ x_n = (m-1)h_1;\ m = 1,...,M;\ y_e = (l-1)h_1; l=1,..,L;\ X = (M-1)h_1,\ Y = (L-1)h_2\right\}.$$

Difference schemes for three variables are constructed similarly.

Let's return to the relation (18), which can be represented in scalar form:

$$\omega_i\frac{\partial u}{\partial t} + \lambda_i\omega_i\frac{\partial u}{\partial x} = \omega_i\left(\frac{\partial u}{\partial t} + \lambda_i\frac{\partial u}{\partial x}\right) = 0,\ i = 1,..,n, \tag{27}$$

where each component of the vector:

$$\frac{\partial u}{\partial t} + \lambda_i\frac{\partial u}{\partial x} \tag{28}$$

is the transfer equation for the corresponding component of the vector u:

$$\frac{\partial u_j}{\partial t} + \lambda_i\frac{\partial u_j}{\partial x} = 0;\ j = 1,..,n.$$

moreover, for each of these equations, it is possible to construct an already known scheme:

$$u_{j,m}^{n+1} = \sum_{\mu,\nu}\alpha_{\mu}^{\gamma}(\tau, h, \lambda_i)u_{j,m+\mu}^{n+\gamma} = \sum_{\mu,\nu}\alpha_{\mu}^{\gamma}(\sigma_i)u_{j,m+\mu}^{n+\gamma}. \tag{29}$$

Next, we replace the vector (28) with its difference approximation:

$$\omega_i\left(\frac{\partial u}{\partial t} + \lambda_i\frac{\partial u}{\partial x}\right) = \left[\omega_i u_m^{n+1} - \sum_{\mu,\nu}\alpha_{\mu}^{\nu}(\sigma_i)\omega_i u_{m+\mu}^{n+\gamma}\right],\ i = 1,..,n, \tag{30}$$

and note that each equation in (30) is the i-th component of the vector:

$$\Omega\frac{\partial u}{\partial t} + \Lambda\Omega\frac{\partial u}{\partial x} = 0;$$

so

$$\Omega\frac{\partial u}{\partial t} + \Lambda\Omega\frac{\partial u}{\partial x} = \begin{cases}\omega_i u_m^{n+1} - \sum_{\mu,\nu}\alpha_{\mu}^{\nu}(\sigma_1)\omega_i\cdot u_{m+\mu}^{n+\gamma} \\ --------------- \\ \omega_n u_m^{n+1} - \sum\alpha_{\mu}^{\gamma}(\sigma_n)\omega_n\cdot u_{m+\mu}^{n+\gamma}\end{cases}. \tag{31}$$

Multiplying the obtained ratio by the matrix Ω^{-1}, obtain a general view of the difference scheme for the numerical solution of a system of hyperbolic equations (17):

$$u_m^{n+1} = \sum_{\mu,\gamma}\Omega^{-1}B_{\mu}^{\gamma}\Omega u_{m+\mu}^{n+\gamma}, \tag{32}$$

where $B_{\mu}^{\gamma} = \mathrm{diag}\{\alpha_{\mu}^{\gamma}(\sigma_i)\}$ is a diagonal matrix with components that are coefficients α^{γ}_{μ}, defining a specific type of scheme.

For example, in the case of considering a difference grid-characteristic scheme earlier, we obtain (see (22), (23), (24)):

$$u_m^{n+1} = \left(\Omega^{-1}E\Omega\right)u_m^n + \frac{\sigma}{2}\left(\Omega^{-1}\Lambda^{+}\Omega\right)\left(u_{m-1}^n - u_m^n\right) - \sigma\left(\Omega^{-1}\Lambda^{+}\Omega\right)\left(u_{m+1}^n - u_m^n\right); \tag{33}$$

or, in another form:

For the Lax-Wendroff scheme (5) obtained earlier for the scalar equation, have:

$$u_m^{n+1} = u_m^n + \frac{\sigma}{2}\left(\Phi_{m-1}^n - \Phi_{m+1}^n\right) + \frac{\sigma^2}{2}\left(\Omega^{-1}|\Lambda|\Omega\right)\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right). \tag{34}$$

Grid-characteristic parametric schemes of higher approximation orders can be presented in the following form:

$$\begin{aligned}u_m^{n+1} &= u_m^n + \frac{\sigma}{2}\left(\Phi_{m-1}^n - \Phi_{m+1}^n\right) + \frac{\sigma^2}{2}\left(\Omega^{-1}\Lambda^2\Omega\right)\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right) + \\ &+ \frac{\sigma}{2}\left(\Omega^{-1}G\Lambda\Omega\right)\left(u_{m-2}^n - 2u_{m-1}^n + u_{m+1}^n - u_{m+2}^n\right) + \frac{\sigma}{2}\left(\Omega^{-1}G|\Lambda|\Omega\right)\left(u_{m-2}^n - 4u_{m-1}^n + 6u_{m+1}^n + u_{m+2}^n\right),\end{aligned} \tag{35}$$

where $G = \mathrm{diag}\ \{g_1(\sigma_1)\}$, $i = 1, ..., n$ is a diagonal parametric matrix.

This scheme can also be represented as a "predictor-corrector" scheme:

$$\tilde{u}_m^n = u_m^n + \frac{\sigma}{2}\left(\Phi_{m-1}^n - \Phi_{m+1}^n\right) + \frac{\sigma}{2}\left(\Omega^{-1}|\Lambda|\Omega\right)\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right),$$

$$u_m^{n+1} = u_m^n + \Omega^{-1}\left(\frac{\sigma^2}{2}\Lambda^2 - \frac{\sigma}{2}|\Lambda|\right)\Omega\left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right) +$$

$$+\Omega^{-1}G\Omega\left[\left(\tilde{u}_m^n - 2\tilde{u}_m^n + \tilde{u}_{m-1}^n\right) - \left(u_{m-1}^n - 2u_m^n + u_{m+1}^n\right)\right].$$

So, at

$$g_i = \frac{\sigma^2{}_i - 1}{6}$$

get a scheme of the 3rd order of accuracy (analogous to the Rusanov scheme).

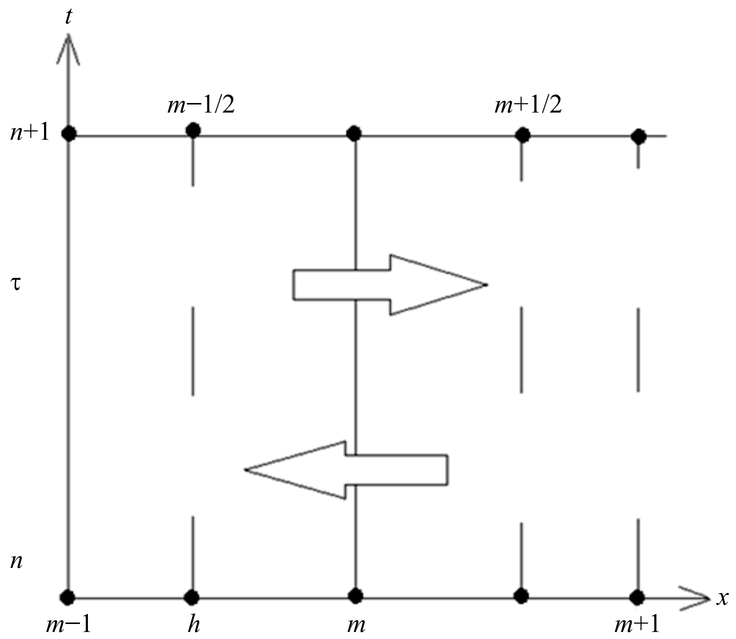

Fig. 2. Predictor-corrector scheme

The grid-characteristic method can also be represented as a flow method, or as an integro-interpolation method, if equation (23) is written in integral form:

$$\iint_s \left(\frac{\partial u}{\partial t} + \frac{\partial \Phi}{\partial x}\right) dt \cdot dx = \oint_\Gamma \left(u dx - \Phi dt\right) = 0, \quad (36)$$

and then the last integral will be approximated:

$$\int_{x-1/2}^{x+1/2} u^{n+1} dx + \int_{t_n}^{t_{n+1}} \Phi_{m+1/2}^{n+1/2} dt - \int_{x_{m-1/2}}^{x_{m+1/2}} u^n dx - \int_{t_n}^{t_{m+1}} \Phi_{m-1/2dt}^{n+1/2} = 0.$$

It follows from the last equality, if the approximation of integrals is carried out by the method of averages, get:

$$u_m^{n+1} h - \Phi_{m+1/2}^{n+1/2} - u_m^n h - \Phi_{m-1/2}^{n+1/2} \tau = 0 \quad (37)$$

(integro-interpolation method),
or:

$$u_m^{n+1} = u_m^n - \sigma\left(\Phi_{m+1/2}^{n+1/2} - \Phi_{m-1/2}^{n+1/2}\right) \quad (38)$$

(flow method).

Note that in the foreign literature these methods are called finite volume methods. It is clear that (40) is a family of numerical methods whose properties depend on the method of calculating flows $\Phi_{m\pm 1/2}^{n+1/2}$; for example, the linear transfer equation (1) flows can be calculated as follows, but with a half-integer upper index):

$$\Phi_{m+1/2}^{n+1/2} = \frac{1}{2}\left\{\frac{1}{2}\left[a\left(u_{m+1/2}^n - u_m^n\right) - |a|\left(u_{m+1}^n - u_m^n\right)\right] + \frac{1}{2}\left[a\left(u_{m+1/2}^{n+1} - u_m^{n+1}\right) - |a|\left(u_{m+1}^{n+1} - u_m^{n+1}\right)\right]\right\},$$

$$\Phi_{m-1/2}^{n+1/2} = \frac{1}{2}\left\{\frac{1}{2}\left[a\left(u_m^n - u_{m-1}^n\right) - |a|\left(u_m^n - u_{m-1}^n\right)\right] + \frac{1}{2}\left[a\left(u_m^{n+1} - u_{m-1}^{n+1}\right) - |a|\left(u_m^{n+1} - u_{m-1}^{n+1}\right)\right]\right\}. \quad (39)$$

Similarly, methods (33), (34), etc. can be presented in streaming form.

In the two-dimensional case, the first-order grid-based approximation method can be presented in the following form [2]:

$$u_{ml}^{n+1} = u_{ml}^{n} + b_{1ml}^{n} + b_{2ml}^{n} + \tau f_{ml}, \qquad (40)$$

where

$$b_{1ml}^{n} = \sigma_1\left[\left(\Omega_1^{-1}\Lambda_1^{+}\Omega_1\right)_{ml}^{n}\left(u_{m-1,l}-u_{ml}\right)^{n} - \left(\Omega_1^{-1}\Lambda_1^{-}\Omega_1\right)_{ml}^{n}\left(u_{m+1,l}-u_{ml}\right)^{n}\right],$$

$$b_{2ml}^{n} = \sigma_2\left[\left(\Omega_2^{-1}\Lambda_2^{+}\Omega_2\right)_{ml}^{n}\left(u_{m-1,l}-u_{ml}\right)^{n} - \left(\Omega_2^{-1}\Lambda_2^{-}\Omega_2\right)_{ml}^{n}\left(u_{m+1,l}-u_{ml}\right)^{n}\right];$$

$$\Lambda_k^{+} = \frac{1}{2}\left(\Lambda_k + |\Lambda_k|\right),\ \Lambda_k^{-} = \frac{1}{2}\left(\Lambda_k - |\Lambda_k|\right),\ |\Lambda_k| = \left\{|\lambda_i^k|\right\},\ \Lambda_k = \left\{\lambda_i^k\right\},$$

$i = 1,..., N$ are the diagonal matrix; λ^k_i are the eigenvalues; $\Omega_k = \{\omega^k_{ij}\}$ are the nonsingular matrices whose rows are linearly independent eigenvalues ω^k_i of the matrix A_k.

For matrices $A_k = \{a^k_{ij}\}$, in the case of a system of equations of mechanics of a deformable solid, we have: $\lambda^k_i = \tilde{v}_k + y^k_i$, $i = 1, ..., 7$, $k = 1,2$.

Using the usual kinematic relations for the components of the symmetric strain rate tensor [8] in a spatially fixed curvilinear orthogonal coordinate system x_1 and x_2, x_3.

$$e_{mn} = \frac{1}{2}\left(\frac{1}{H_n}\frac{\partial \upsilon_m}{\partial x_n} + \frac{1}{H_m}\frac{\partial \upsilon_n}{\partial x_m}\right) + \frac{1}{H_n}\left[\delta_{mn}\left(\sum_{s=1}^{3}\frac{\upsilon_S}{H_S}\frac{\partial H_m}{\partial x_S}\right) - \frac{1}{2H_n}\left(\upsilon_m\frac{\partial H_m}{\partial x_n} + \frac{\partial H_m}{\partial x_m}\right)\right],$$

$m, n = 1, 2, 3.$

and choosing the defining relations in the form:

$$\frac{d\sigma_{ij}}{d\tilde{t}} = \frac{d\sigma_{ij}}{d\tilde{t}} + \frac{\upsilon_1}{H_1}\frac{d\sigma_{ij}}{dx_1} + \frac{\upsilon_2}{H_2}\frac{d\sigma_{ij}}{dx_2} + \frac{\upsilon_3}{H_3}\frac{d\sigma_{ij}}{dx_3} = \sum_{m,n=1}^{3} q_{ijmn} emn, ij = 1, 2, 3,$$

write a closed system of two-dimensional unsteady equations in the form:

$$\frac{\partial u}{\partial t} + A_1\frac{\partial u}{\partial x_1} + A_2\frac{\partial u}{\partial x_2} = f.$$

Strictly speaking, instead of a substantial derivative $d\sigma_{ij} / dt$ the so-called Yaumann derivative should be used for the deviator components of a stress tensor of the type:

$$\frac{\tilde{d}S}{dt} = \frac{dS}{dt} - \sum_{k=1}^{3}\left(S_{ik}\omega_{jk} + S_{jk}\omega_{ik}\right),$$

$$\omega_{ij} = \frac{1}{2}\left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right), \quad S_{ij} = \sigma_{ij} - \delta_{ij}\frac{1}{3}\sum_{k=1}^{3}\sigma_{kk},$$

which ensures zero rate of change of the stressed state of the particle during its rotation as a rigid whole.

Here the symbol $u = \{v_1, v_2, \sigma_{11}, \sigma_{22}, \sigma_{33}, T\}$ is a vector of the desired variables, including the components of the velocity vector v_1 and v_2 (along the axes — x_1, x_2 respectively), nonzero components σ_{ij} of the symmetric stress tensor and temperature T; $f(t, x_1, x_2, u) = \{f_1, f_2, f_{11}, f_{12}, f_{22}, f_{33}, f_T\}$ is the vector of the right parts with the following components in a curved orthogonal coordinate system x_1, x_2, x_3:

$$f_1 = F_1 + \frac{1}{\rho}\left[\frac{(\sigma_{11} - \sigma_{22})}{H_1 H_2} + \frac{\partial H_2}{\partial x_1} + \frac{(\sigma_{11} - \sigma_{33})}{H_1 H_3} + \frac{\partial H_{12}}{H_2}\left(\frac{2}{H_1}\frac{\partial H_1}{\partial x_2} + \frac{1}{H_3}\frac{\partial H_3}{\partial x_2}\right)\right] - \frac{v_2}{H_1 H_2}\left(v_1\frac{\partial H_1}{\partial x_2} + v_2\frac{\partial H_2}{\partial x_1}\right),$$

$$f_2 = F_2 + \frac{1}{\rho}\left[\frac{(\sigma_{22} - \sigma_{33})}{H_2 H_3} + \frac{\partial H_3}{\partial x_2} + \frac{(\sigma_{22} - \sigma_{11})}{H_2 H_1} + \frac{\partial H_{12}}{H_1}\left(\frac{2}{H_2}\frac{\partial H_2}{\partial x_1} + \frac{1}{H_3}\frac{\partial H_3}{\partial x_1}\right)\right] - \frac{v_1}{H_2 H_1}\left(v_2\frac{\partial H_2}{\partial x_1} + v_1\frac{\partial H_1}{\partial x_2}\right),$$

$$f_{ij} = \frac{q_{ij11} v_2}{H_1 H_2}\frac{\partial H_2}{\partial x_2} - \frac{(q_{ij12} - q_{ij21})}{2 H_1 H_2}\left(v_1\frac{\partial H_1}{\partial x_2} + v_2\frac{\partial H_2}{\partial x_1} + \right) + \frac{q_{ij22} v_2}{H_1 H_2}\frac{\partial H_2}{\partial x_1} + \frac{q_{ij33}}{H_3}\left(\frac{v_1}{H_1}\frac{\partial H_3}{\partial x_1} + \frac{v_2}{H_2}\frac{\partial H_3}{\partial x_2}\right) - \frac{\sigma_{ij}\gamma Q}{\rho c},$$

$$f_T = \frac{1}{\rho c}\left[\frac{\sigma_{11} v_2}{H_1 H_2}\frac{\partial H_1}{\partial x_2} - \frac{\sigma_{22}}{H_1 H_2} + \left(v_1\frac{\partial H_1}{\partial x_2} + v_2\frac{\partial H_2}{\partial x_1}\right) + \frac{\sigma_{ij22} v_2}{H_1 H_2}\frac{\partial H_2}{\partial x_1} + \frac{\sigma_{33}}{H_3}\left(\frac{v_1}{H_1}\frac{\partial H_3}{\partial x_1} + \frac{v_2}{H_2}\frac{\partial H_3}{\partial x_2}\right) + Q\right].$$

Here $F1$, $F2$ are the components of the vector of mass forces, T s the internal energy for a thermoelastic medium, c is the specific heat capacity of the material, T is the temperature, Q is the volumetric density of heat sources, H_i, i =1, 2, 3 are the Lamb coefficients characterizing the selected orthogonal curved coordinate system, ρ is the density, defined by the equation of state of a solid, for example:

$$\ln\frac{\rho}{\rho_0} = -(3K)^{-1}\sum_{i=1}^{3}\sigma_{ii} - 3\alpha T,$$

ρ_0 is the density of the material in the undeformable state, K is the volume compression ratio. In accordance with the two-dimensionality of the deformable state assumed here, there are no displacements of material points in the x_3 direction and $\sigma_{13} = \sigma_{23} = v_3 = 0$, $\partial/\partial x_3 = 0$. Matrices $A_k = \{a_{ij}^k\}, k = 1,2,\ i = 1,\ldots,7,$ at the same time have the form:

$$A_1 = \frac{1}{H_1}\begin{vmatrix} v_1 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\ 0 & v_1 & 0 & -1/\rho & 0 & 0 & 0 \\ -q_{1111} & -(q_{1112}+q_{1121})/2 & v_1 & 0 & 0 & 0 & 0 \\ q_{1211} & -(q_{1212}+q_{1221})/2 & 0 & v_1 & 0 & 0 & 0 \\ q_{2211} & -(q_{2211}+q_{2221})/2 & 0 & 0 & v_1 & 0 & 0 \\ q_{3311} & -(q_{3312}+q_{3321})/2 & 0 & 0 & 0 & v_1 & 0 \\ -\sigma_{11}/\rho_1 & -\sigma_{11}/\rho_1 & 0 & 0 & 0 & 0 & v_1 \end{vmatrix},$$

$$A_2 = \frac{1}{H_2}\begin{vmatrix} v_2 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\ 0 & v_2 & 0 & -1/\rho & 0 & 0 & 0 \\ -(q_{1112}+q_{1121})/2 & -q_{1111} & v_2 & 0 & 0 & 0 & 0 \\ -(q_{1212}+q_{1221})/2 & -q_{1211} & 0 & v_2 & 0 & 0 & 0 \\ -(q_{2211}+q_{2221})/2 & -q_{2211} & 0 & 0 & v_2 & 0 & 0 \\ q_{3311} & -(q_{3312}+q_{3321})/2 & 0 & 0 & 0 & v_2 & 0 \\ -\sigma_{12}/\rho_2 & -\sigma_{22}/\rho_2 & 0 & 0 & 0 & 0 & v_2 \end{vmatrix}.$$

For the Prandtl-Reiss model adopted in this paper, the components of the fourth-rank qiikl tensor q_{iikl} of an elastic-plastic material have the form [8]:

$$q_{ijkl} = \lambda\,\delta_{ij}\delta_{kl} - \frac{\gamma\sigma_{ij}\delta_{ij}}{\rho c} + \mu\left(\delta_{ik}\delta_{jl} - \delta_{jk}\delta_{il}\right) - \frac{I\mu\mu_{ij}S_{kl}}{k},$$

where λ, μ are the Lyame parameters, k is the shear yield strength, δ_{mn} are Kronecker symbols, stress tensor deviator:

$$S_{mn} = \sigma_{mn} - \delta_{mn}\frac{1}{3}\sum_{S=1}^{3}\sigma_{ss},$$

γ = (3λ + 2μ) α (α is the coefficient of linear expansion of the material during heating).

I is determined from the Mises plasticity condition:

$$I = \begin{cases} 0 \text{ at } & S = S_{11}^2 + S_{22}^2 + S_{33}^2 + S_{12}^2 < 2k^2, \\ 1 \text{ at } & S \ge 2k^2. \end{cases}$$

The defining relations at $I = 0$ are the usual “Hooke’s law” for elastic isotropic materials. For $y = 0$, $Q = 0$, the first 6 equations do not include temperature and they can be solved independently of the energy equation.

$$y_1^k = -y_7^k = \left[a_k + \left(a_k^2 - 4d_k\right)^{1/2}\right]^{1/2},\quad y_2^k = -y_6^k = \left[a_k - \left(a_k^2 - 4d_k\right)^{1/2}\right]^{1/2},\quad y_3^k = -y_4^k = y_5^k = 0.$$

$$a_k = a_{13}^k a_{31}^k + a_{14}^k a_{41}^k + a_{24}^k a_{42}^k + a_{25}^k a_{52}^k,$$

$$d_k = a_{13}^k a_{24}^k\left(\left(a_{31}^k a_{42}^k - a_{32}^k a_{41}^k\right)\right) + a_{14}^k a_{25}^k \times \left(\left(a_{41}^k a_{32}^k - a_{42}^k a_{31}^k\right)\right) + a_{13}^k a_{25}^k\left(\left(a_{31}^k a_{32}^k - a_{32}^k a_{31}^k\right)\right),$$

$k = 1,2,$

$$\mathbf{\Omega}_1 = \begin{Vmatrix} \omega_1^1 \\ \omega_2^1 \\ \omega_3^1 \\ \omega_4^1 \\ \omega_5^1 \\ \omega_6^1 \\ \omega_7^1 \end{Vmatrix} = \begin{Vmatrix} 1 & \omega_{12}^1 & \omega_{13}^1 & \omega_{14}^1 & \omega_{15}^1 & 0 & 0 \\ \omega_{21}^1 & 1 & \omega_{23}^1 & \omega_{24}^1 & \omega_{25}^1 & 0 & 0 \\ 0 & 0 & \omega_{32}^1 & \omega_{34}^1 & 1 & 0 & 0 \\ 0 & 0 & \omega_{43}^1 & \omega_{44}^1 & 0 & 1 & 0 \\ 0 & 0 & \omega_{53}^1 & \omega_{54}^1 & 0 & 0 & 1 \\ \omega_{21}^1 & 1 & -\omega_{23}^1 & -\omega_{24}^1 & -\omega_{25}^1 & 0 & 0 \\ 1 & \omega_{12}^1 & -\omega_{13}^1 & -\omega_{14}^1 & -\omega_{15}^1 & 0 & 0 \end{Vmatrix},$$

$$\mathbf{\Omega}_2 = \begin{Vmatrix} \omega_{11}^2 & 1 & \omega_{13}^2 & \omega_{14}^2 & \omega_{15}^2 & 0 & 0 \\ 1 & \omega_{22}^2 & \omega_{23}^2 & \omega_{24}^{21} & \omega_{25}^2 & 0 & 0 \\ 0 & 0 & 1 & \omega_{34}^2 & \omega_{35}^2 & 0 & 0 \\ 0 & 0 & 0 & \omega_{44}^2 & \omega_{45}^2 & 1 & 0 \\ 0 & 0 & 0 & \omega_{54}^2 & \omega_{55}^2 & 0 & 1 \\ 1 & \omega_{22}^2 & -\omega_{23}^2 & -\omega_{24}^2 & -\omega_{25}^2 & 0 & 0 \\ \omega_{11}^2 & 1 & -\omega_{13}^2 & -\omega_{14}^2 & -\omega_{15}^2 & 0 & 0 \end{Vmatrix}.$$

Here:

$$\omega_{12}^1 = \frac{\left(y_1^1 - \alpha_1^1\right)}{\alpha_2^1}, \qquad \omega_{21}^1 = \frac{\alpha_2^1}{\left(y_2^1\right)^2 - \alpha_1^1},$$

$$\omega_{11}^1 = \frac{\alpha_2^2}{\left(y_1^2\right)^2 - \alpha_1^2}, \quad \omega_{12}^1 = \frac{\left(y_2^2\right) - \alpha_1^2}{\alpha_2^2},$$

$$\alpha_i^k = a_{13}^k a_{31}^k + a_{14}^k a_{41}^k, \qquad \alpha_2^k = a_{24}^k a_{41}^k + a_{25}^k a_{51}^k, k = 1,2,$$

$$\omega_{i3}^1 = \frac{a_{13}^k \omega_{i1}^k}{y_i^k}, \qquad \omega_{i4}^1 = \frac{a_{14}^k \omega_{i1}^k + a_{24}^k \omega_{i2}^k}{y_i^k},$$

$$\omega_{i5}^k = \frac{a_{25}^k \omega_{12}^k}{y_i^k}, i = 1,2,$$

$$\omega_{i3}^k = \frac{a_{42}^k \beta_{i1}^1 - a_{41}^1 \beta_{i2}^1}{a_{32}^1 \beta_{i2}^1 - a_{41}^1 \beta_{i2}^1}, \quad \omega_{i4}^k = \frac{a_{32}^k \beta_{i1}^1 - a_{31}^1 \beta_{i2}^1}{a_{31}^1 \beta_{42}^1 - a_{32}^1 \beta_{41}^1},$$

$$\omega_{i4}^2 = \frac{a_{52}^2 \beta_{i1}^2 - a_{51}^2 \beta_{i2}^2}{a_{51}^2 \beta_{i2}^2 - a_{52}^1 \beta_{41}^2}, \quad \omega_{i5}^k = \frac{a_{42}^2 \beta_{i1}^2 - a_{41}^2 \beta_{i2}^2}{a_{52}^2 \beta_{i1}^2 - a_{51}^1 \beta_{42}^2},$$

$$\beta_{ij}^1 = a_{5j}^1 \omega_{i5}^1 + a_{6j}^1 \omega_{i6}^1 + a_{7j}^1 \omega_{i7}^1,$$

$$\beta_{ij}^2 = a_{3j}^2 \omega_{i3}^2 + a_{6j}^2 \omega_{i6}^2 + a_{7j}^2 \omega_{i7}^2, \ \ j = 1,2, \ \ i = 3,4,5,$$

$$\mathbf{\Omega}_k^{-1} = \left\{ p_{ij}^k \right\}.$$

in this case, for inverse matrices have:

$$\mathbf{\Omega}^{-1}{}_k = \begin{Vmatrix} p_{11}^k & p_{12}^k & 0 & 0 & 0 & p_{12}^k & p_{11}^k \\ p_{21}^k & p_{22}^k & 0 & 0 & 0 & p_{22}^k & p_{21}^k \\ p_{31}^k & p_{32}^k & p_{33}^k & 0 & 0 & -p_{32}^k & -p_{31}^k \\ p_{41}^k & p_{42}^k & p_{43}^k & 0 & 0 & -p_{42}^k & -p_{41}^k \\ p_{51}^k & p_{52}^k & p_{53}^k & 0 & 0 & -p_{52}^k & -p_{51}^k \\ p_{61}^k & p_{62}^k & p_{63}^k & 1 & 0 & -p_{62}^k & -p_{61}^k \\ p_{71}^k & p_{72}^k & p_{73}^k & 0 & 1 & -p_{72}^k & -p_{71}^k \end{Vmatrix},$$

where

$$p_{11}^1 = p_{22}^1 = \frac{(\Delta_2^1)}{2}, \qquad p_{12}^1 = -\frac{\omega_{12}^1}{2\Delta_2^1}, \qquad p_{21}^1 = -\frac{\omega_{21}^1}{2\Delta_2^1}, \qquad \Delta_2^1 = 1 - \omega_{12}^1 \omega_{21}^1.$$

$$p_{31}^1 = \frac{\omega_{25}^1 \omega_{34}^1 - \omega_{14}^1 \omega_{25}^1}{2\Delta_1^1}, \qquad p_{32}^1 = \frac{\omega_{14}^1 - \omega_{15}^1 \omega_{34}^1}{2\Delta_1^1},$$

$$p_{33}^1 = \frac{\omega_{15}^1 \omega_{24}^1 - \omega_{14}^1 \omega_{25}^1}{2\Delta_1^1}, \qquad p_{41}^1 = \frac{\omega_{23}^1 - \omega_{25}^1 \omega_{33}^1}{2\Delta_1^1},$$

$$p^1_{42} = \frac{\omega^1_{15}\omega^1_{33} - \omega^1_{13}}{2\Delta^1_1}, \qquad p^1_{43} = \frac{\omega^1_{25}\omega^1_{13} - \omega^1_{15}\omega^1_{23}}{\Delta^1_1},$$

$$p^1_{51} = \frac{\omega^1_{24}\omega^1_{33} - \omega^1_{23}\omega^1_{34}}{2\Delta^1_1}, \quad p^1_{52} = \frac{\omega^1_{13}\omega^1_{24} - \omega^1_{14}\omega^1_{33}}{2\Delta^1_1},$$

$$p^1_{53} = \frac{\omega^1_{14}\omega^1_{23} - \omega^1_{13}\omega^1_{24}}{\Delta^1_1},$$

$$p^1_{i1} = \left[\omega^1_{i-2,3}\left(\omega^1_{24} - \omega^1_{25}\omega^1_{34}\right) - \omega^1_{i-2,4}\left(\omega^1_{23} - \omega^1_{25}\omega^1_{33}\right)\right]\left(2\Delta^1_1\right)^{-1},$$

$$p^1_{i2} = \left[\omega^1_{i-2,4}\left(\omega^1_{13} - \omega^1_{15}\omega^1_{33}\right) - \omega^1_{i-2,3}\left(\omega^1_{14} - \omega^1_{15}\omega^1_{34}\right)\right]\left(2\Delta^1_1\right)^{-1},$$

$$p^1_{i3} = \left[\omega^1_{i-2,3}\left(\omega^1_{14}\omega^1_{25} - \omega^1_{15}\omega^1_{24}\right) - \omega^1_{i-2,4}\left(\omega^1_{13}\omega^1_{25} - \omega^1_{15}\omega^1_{23}\right)\right]\left(\Delta^1_1\right)^{-1},$$

$$\Delta^1_1 = \omega^1_{15}\left(\omega^1_{24}\omega^1_{33} - \omega^1_{23}\omega^1_{34}\right) + \omega^1_{25}\left(\omega^1_{13}\omega^1_{34} - \omega^1_{14}\omega^1_{33}\right) + \left(\omega^1_{14}\omega^1_{23} - \omega^1_{13}\omega^1_{24}\right).$$

And, similarly:

$$p^2_{43} = \frac{\omega^2_{23}\omega^2_{23} - \omega^2_{13}\omega^2_{25}}{\Delta^2_1}, \quad p^1_{54} = \frac{\omega^2_{23}\omega^2_{34} - \omega^2_{24}}{2\Delta^2_1},$$

$$p^2_{52} = \frac{\omega^2_{14} - \omega^2_{13}\omega^2_{34}}{2\Delta^2_1}, \qquad p^1_{53} = \frac{\omega^2_{13}\omega^2_{24} - \omega^2_{23}\omega^2_{14}}{\Delta^2_1},$$

$$p^2_{i1} = \left[\omega^2_{i-2,3}\left(\omega^2_{24} - \omega^2_{25}\omega^2_{34}\right) - \omega^2_{i-2,4}\left(\omega^2_{25} - \omega^2_{23}\omega^2_{35}\right)\right]\left(2\Delta^2_1\right)^{-1},$$

$$p^2_{i2} = \left[\omega^2_{i-2,4}\left(\omega^2_{15} - \omega^2_{13}\omega^2_{35}\right) - \omega^2_{i-2,5}\left(\omega^2_{14} - \omega^2_{13}\omega^2_{34}\right)\right]\left(2\Delta^1_2\right)^{-1},$$

$$p^1_{i3} = \left[\omega^2_{i-2,4}\left(\omega^2_{13}\omega^2_{25} - \omega^2_{15}\omega^2_{23}\right) - \omega^2_{i-2,5}\left(\omega^2_{14}\omega^2_{23} - \omega^2_{13}\omega^2_{24}\right)\right]\left(\Delta^2_1\right)^{-1},$$

$$i = 6, 7,$$

$$\Delta^2_1 = \omega^2_{13}\left(\omega^2_{24}\omega^2_{35} - \omega^2_{25}\omega^2_{34}\right) + \omega^2_{23}\left(\omega^2_{15}\omega^2_{34} - \omega^2_{14}\omega^2_{35}\right) + \left(\omega^2_{14}\omega^2_{25} - \omega^2_{15}\omega^2_{24}\right).$$

When constructing calculation formulas on the boundaries of a rectangular (in coordinates t, x_1, x_2) integration domain, we limit ourselves to considering only the upper ($x_2 = 0$) and lower ($x_2 = 1$) boundaries, bearing in mind that the remaining boundaries ($x_1 = 0, x_{1*}$) are often the plane (or axis) of symmetry or periodicity of the solution, or are chosen in such a way that perturbations do not reach these boundaries during the time under consideration $t \le t_1$. Generalization to the case of more complex conditions at the borders $x_1 = 0$ does not present fundamental difficulties and is similar to the one discussed below.

Multiplying scalar by eigenvectors $(\omega^2_i)^n_{ml}$, obtain the relations:

$$\left(\omega^2_i\right)^n_{ml} u^{n+1}_{ml} = B^2_i = \left(\omega^2_i\right)^n_{ml}\left(u^n_{ml} + \tau f^n_{ml}\right) \pm$$

$$\pm \frac{\tau}{h_2}\left(\lambda^2_i\right)^n_{ml}\left(\omega^2_i\right)^n_{ml}\left(u_{m,l\mp 1} + u^n_{ml}\right), i = 1,...,7,$$

compatibility conditions approximating with the first order of accuracy along the intersection lines of the characteristic surfaces of the system and the coordinate plane $x_1 = x_{1m}$ (with equations $dx_2 = \lambda^2_i dt$):

$$\omega^2_i u_t + \lambda^2_i \omega^2 u_{x2} = \omega^2_i (f - A_1 u_{x2}), \ i = 1,.., 7.$$

As is known, the number of boundary conditions for a hyperbolic system of equations of the type is determined by the number of negative (positive) eigenvalues of the matrix Au at the upper (respectively, lower) boundary of the integration domain. In the problems considered below, there is a situation $\lambda^2_7 < \lambda^2_6 < 0$, at the upper boundary $x_1 = 0$, and at the lower boundary $x_2 = 0$ respectively $\lambda^2_1 > \lambda^2_1 > 0$ therefore, two boundary conditions are required at each of these boundaries, which look like

$$\Phi_i(t, x_1, u_1,..., u_7) = 0, \ i = 1, 2 \text{ при } x_2 = 0,$$

$$\Phi_i(t, x_1, u_1,..., u_7) = 0, \ i = 167 \text{ при } x_2 = 1,$$

причем необходимо, чтобы $\det \Omega_- \ne 0$, $\det \Omega_+ \ne 0$, where, respectively, $\Omega_- = \left\| \overline{\omega}_1 \overline{\omega}_2 \omega_3 ... \omega_7 \right\|^T$, $\Omega_+ = \left\| \omega_1 ... \omega_5 \overline{\omega}_6 \overline{\omega}_7 \right\|^T$.

Where $\overline{\omega}_i = \{\partial\phi_i / \partial u_1, ..., \partial\varphi_i / \partial u_7\}, i = 1,2,6,7$, a $i = 1,...7$ are the eigenvectors of the A_2 matrix. For the problems considered below, the boundary conditions were chosen semi-linear and after their approximation had the form:

$$\phi_i = \overline{\omega}_i \left(t^{n+1}, x_{1m}\right) u_{ml}^{n+1} - g_i \left(t^{n+1}, x_{1m}\right) = 0,$$

$$i = 1,2 \text{ при } x_2 = 0,\ i = 6,7 \text{ при } x_2 = 1.$$

Consider a possible splitting scheme for the case of two variables:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} + B\frac{\partial u}{\partial y} = 0:$$

$$\beta_1\left(\frac{\partial u}{\partial t} + \beta_1^{-1} A\frac{\partial u}{\partial x}\right) + \beta_2\left(\frac{\partial u}{\partial t} + \beta_2^{-1} B\frac{\partial u}{\partial y}\right) = 0,$$

$$\beta_1 + \beta_2 = 1,\ \beta \geq 0,\ \beta_2 \geq 0;\ u_{ml}^{n+1} = \beta_1 u_{1ml}^{n+1} + \beta_2 u_{2ml}^{n+1},$$

where u_{1ml}^{n+1} and u_{2ml}^{n+1} are from the numerical solution of two one-dimensional systems of equations:

$$\frac{\partial u}{\partial t} + \beta_1^{-1} A\frac{\partial u}{\partial x} = 0 \text{и} \frac{\partial u}{\partial t} + \beta_2^{-1} A\frac{\partial u}{\partial y} = 0.$$

In the case of a spatial dynamic problem: $\frac{\partial u}{\partial t} + \sum_{k=1}^{3} A_k \frac{\partial u_k}{\partial x_k} = 0$

the matrices have the form: $r_{ijkl} = \frac{1}{2}\left(q_{ijkl} + q_{ijkl}\right), l \neq k;$

$u = \{v_1, v_2, v_3, \sigma_{11}, \sigma_{12}, \sigma_{13}, \sigma_{22}, \sigma_{23}, \sigma_{33}, U\}$ is the vector column of the desired values.

$$A_1 = \begin{Vmatrix} v_1 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_1 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_1 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\ -q_{1111} & -r_{1112} & 0 & v_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -q_{1211} & -r_{1212} & 0 & 0 & v_1 & 0 & 0 & 0 & 0 & 0 \\ -q_{1311} & -r_{2212} & 0 & 0 & 0 & v_1 & 0 & 0 & 0 & 0 \\ -q_{2211} & -r_{1312} & 0 & 0 & 0 & 0 & v_1 & 0 & 0 & 0 \\ -q_{2311} & -r_{3212} & 0 & 0 & 0 & 0 & 0 & v_1 & 0 & 0 \\ -q_{3311} & -r_{3312} & 0 & 0 & 0 & 0 & 0 & 0 & v_1 & 0 \\ -\sigma_{11}/\rho & -\sigma_{12}/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_1 \end{Vmatrix},$$

$$A_2 = \begin{Vmatrix} v_2 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_2 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_2 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\ -r_{1112} & -q_{1122} & 0 & v_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ -r_{1212} & -q_{1232} & 0 & 0 & v_2 & 0 & 0 & 0 & 0 & 0 \\ -r_{1312} & -q_{1322} & 0 & 0 & 0 & v_2 & 0 & 0 & 0 & 0 \\ -r_{1222} & -q_{2222} & 0 & 0 & 0 & 0 & v_2 & 0 & 0 & 0 \\ -r_{2312} & -q_{2322} & 0 & 0 & 0 & 0 & 0 & v_2 & 0 & 0 \\ -r_{3312} & -q_{3322} & 0 & 0 & 0 & 0 & 0 & 0 & v_2 & 0 \\ -\sigma_{12}/\rho & -\sigma_{22}/\rho & 0 & 0 & 0 & 0 & 0 & 0 & 0 & v_2 \end{Vmatrix},$$

$$A_2 = \begin{Vmatrix} v_3 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_3 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & v_3 & 0 & 0 & -1/\rho & 0 & 0 & 0 & 0 \\ -r_{1113} & -r_{1123} & -q_{1133} & v_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ -r_{1213} & -r_{1223} & -q_{1233} & 0 & v_3 & 0 & 0 & 0 & 0 & 0 \\ -r_{1313} & -r_{1323} & -q_{1333} & 0 & 0 & v_3 & 0 & 0 & 0 & 0 \\ -r_{2213} & -r_{2223} & -q_{2233} & 0 & 0 & 0 & v_3 & 0 & 0 & 0 \\ -r_{2313} & -r_{2323} & -q_{2333} & 0 & 0 & 0 & 0 & v_3 & 0 & 0 \\ -r_{3313} & -r_{3323} & -q_{3333} & 0 & 0 & 0 & 0 & 0 & v_3 & 0 \\ -\sigma_{13}/\rho & -\sigma_{23}/\rho & -\sigma_{33}/\rho & 0 & 0 & 0 & 0 & 0 & 0 & v_3 \end{Vmatrix}.$$

One of the possible grid-characteristic schemes for the numerical solution of the reduced system can be represented as:

$$u_{mlp}^{n+1} = u_{mlp}^{n+1} - \sigma_1\Omega_1^{-1}\Lambda_1\Delta_m u + \sigma_1^{\gamma}\Omega_1\left|\Lambda_1\right|^{\gamma}\Omega_1\Delta_m^2 u - \sigma_2\Omega_2^{-1}\Lambda_1\Delta_l u +$$
$$+\sigma_2^{\gamma}\Omega_2\left|\Lambda_2\right|^{\gamma}\Omega_2\Delta_l^2 u - \sigma_3\Omega_3^{-1}\Lambda_1\Delta_p u + \sigma_3^{\gamma}\Omega_3\left|\Lambda_3\right|^{\gamma}\Omega_2\Delta_p^2 u,$$
$$u_m = 0,5\left(u_{m+1,l,p}^{n} - u_{m-1,l,p}^{n}\right),$$
$$u_l = 0,5\left(u_{m,l,p+1}^{n} - u_{m,l,p-1}^{n}\right),$$
$$u_m^2 = 0,5\left(u_{m+1,l,p}^{n} - 2u_{mlp}^{n} + u_{m-1,l,p}^{n}\right),$$
$$u_l^2 = 0,5\left(u_{m,l+1,p}^{n} - 2u_{mlp}^{n} + u_{m,l-1,p}^{n}\right),$$
$$u_p^2 = 0,5\left(u_{m,l,p+1}^{n} - 2u_{mlp}^{n} + u_{m,l,p-1}^{n}\right),$$

h_k (τh_k — шаги пare the steps by time and three coordinates; k = 1,2,3), or

$$u_{mlp}^{n+1} = u_{mlp}^{n} + \sigma_1\left[\left(\Omega_1^{-1}\Lambda_1^{+}\Delta_m\right)_{mlp}\Delta_m^{-}u - \left(\Omega_1^{-1}\Lambda_1^{+}\Omega_1\right)_{mlp}\Delta_m^{+}u\right]^n +$$
$$+\sigma_2\left[\left(\Omega_2^{-1}\Lambda_2^{+}\Delta_m\right)_{mlp}\Delta_m^{-}u - \left(\Omega_2^{-1}\Lambda_2^{+}\Omega_2\right)_{mlp}\Delta_m^{+}u\right]^n + \sigma_3\left[\left(\Omega_3^{-1}\Lambda_3^{+}\Delta_m\right)_{mlp}\Delta_m^{-}u - \left(\Omega_3^{-1}\Lambda_3^{+}\Omega_3\right)_{mlp}\Delta_m^{+}u\right]^n,$$
$$\Delta_l^{\pm} = u_{m,l\pm1,p} - u_{mlp},$$
$$\Delta_p^{\pm} = u_{m,l,p\pm1} - u_{mlp}.$$

The scheme has the first order of accuracy at $\gamma = 1$ which was implemented in the work), the second at $\gamma = 2$; the "damping" (viscous) term is clearly highlighted, which gives the scheme stability and positive (according to Friedrichs [5]) certainty (or monotony — in the one-dimensional case); when implemented, there is no need to find matrices $\Lambda^{\pm}_k$; the scheme is easy to hybridize (i. e., for example, to assume $\gamma = 1$ in areas with large gradients of flow parameters and $\gamma = 2$ in "smooth" areas). The advantage of recording a grid-characteristic scheme is to reduce the number of arithmetic operations.

Discussion and conclusions. The construction of a numerical algorithm at boundary points is described in detail in [8] for the two-dimensional case. When numerically solving a three-dimensional problem, the construction is performed similarly; for example, in the case of upper and lower bounds, after scalar multiplication of the scheme by eigenvectors $(\omega^3_i)^n_{mlp}$ obtain the following relations:

$$\left(\omega_i^3\right)_{mlp}^n u_{mlp}^{n+1} = \left(\omega_i^3\right)_{mlp}^n u_{mlp}^{n} + \sigma_1\left\{\left[\left(\Omega_1^{-1}\Lambda_1^{+}\Omega_1\right)_{mlp}\Delta_m^{-}u - \left(\Omega_2^{-1}\Lambda_2^{+}\Omega_2\right)_{mlp}\Delta_m^{+}u\right]^n +\right.$$
$$\left.+\sigma_2\left[\left(\Omega_2^{-1}\Lambda_2^{+}\Omega_2\right)_{mlp}\Delta_l^{-}u - \left(\Omega_2^{-1}\Lambda_2^{-}\Omega_2\right)_{mlp}\Delta_t^{+}u\right]^n\right\} \pm \sigma_3\left(\lambda_i^3\right)_{mlp}^n\left(\omega_i^3\right)_{mlp}^n\Delta_t^{\pm}u;\quad i = 1,\dots,I,$$

compatibility conditions approximating with the first order of accuracy

$$\omega_i^3 u_t + \lambda_i^3\omega_t^3 u'_{\eta_3} = -\omega_i^3\left(A_1 u'_{\eta_1} + A_2 u'_{\eta_2}\right).$$

In the case of a one-dimensional system of gas dynamics equations:

$$\frac{\partial u}{\partial t} + A\frac{\partial u}{\partial x} = 0$$

the vector of the desired functions and the matrix A have the following form [2]:

$$u = \begin{Bmatrix}\rho\\ u\\ \varepsilon\end{Bmatrix},\quad A = \begin{Bmatrix} u & p & 0\\ \rho^{-1}\dfrac{\partial p}{\partial s} & u & \rho^{-1}\dfrac{\partial p}{\partial \varepsilon}\\ 0 & p/\rho & u\end{Bmatrix},$$

where ρ is the density, u is the velocity, ε is the density of the internal energy of the gas.

The actual values of the matrix have the form:

$$\lambda_1 = u + c;\quad \lambda_2 = u;\quad \lambda_3 = u - c,$$

where c is the speed of sound in a gas, the matrices from the eigenvectors, in fact, will be:

$$\Omega = \begin{Bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{Bmatrix} = \begin{Bmatrix} \frac{\partial p}{\partial \rho} & \rho c & \frac{\partial p}{\partial \varepsilon} \\ p & 0 & -\rho^2 \\ p & -\rho e & \frac{\partial p}{\partial \rho} \end{Bmatrix}.$$

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Conflict of interest statement

The authors do not have any conflict of interest.

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