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Adaptive modified alternating-triangular method*

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The paper covers the development of adaptive methods for solving convection-diffusion problems. For this class of problems a version of minimal correction adaptive modified alternating-triangular method was developed, as well the convergence rate of the method was analytically estimated. Numerical results have been presented for solving real hydrodynamical problem in shallow water using parallel algorithm of developed method.

Keywords: wave problem; grid equations; adaptive modified alternating triangular iterative method

Introduction. Often in applied problems, for example, in the mathematical modeling of hydrodynamics, heat and mass transfer, geofiltration, population dynamics and other processes, it becomes necessary to solve equations of the convection-diffusion type. In the case of using implicit schemes and schemes with weights, such problems lead to linear algebraic equations with a nonselfadjoint operator. One approach to solving such problems is the Gaussian symmetrization method [3]. The disadvantage of this method is the quadratic increase in the number of conditionality of the operator of the problem.

In the class of two-layer iterative methods one of the most successful is the alternate-triangular method offered by A.A. Samarskiy is [1]. Later, Academician A.N. Konovalov developed the adaptive version of the ATM [2]. In [4], a modified alternate-triangular iteration method for solving the Dirichlet problem for the Poisson equation is described, as well as a description of the simple iteration method in the case of a nonselfadjoint operator. In [8], a technique for increasing the rate of convergence of ATM with a priori information by improving the spectral estimates of the pre-imposed operator was demonstrated. In this paper, a variant of a modified iterative alternating-triangular method of minimal corrections for the solution of grid equations with a nonselfadjoint operator is developed, and estimates of the rate of convergence are performed. It should also be noted that the proposed version of the method of minimum corrections is a separate method. The developed algorithms were applied to solve the grid equations obtained as a result of approximation of the three-dimensional mathematical model of hydrodynamics of shallow water bodies [9-16]. A parallel implementation of the adaptive modified iterative alternating-triangular method of minimal corrections is described [17-22].

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Modified alternate-triangular iterative method. In a finite-dimensional Hilbert space H we consider the problem of finding the solution of the operator equation:

$$Ax = f, \quad A: H \to H, \tag{1}$$

where A is a linear, positive definite operator ($A > 0$). To find the problem (1) we will use an implicit iteration process:

$$B\frac{x^{m+1} - x^m}{\tau} + Ax^m = f, \quad B: H \to H. \tag{2}$$

In equation (2) m is the iteration number, $\tau > 0$ is the iteration parameter, and B is the invertible operator. Call of the operator B in (2) must be substantially simpler than the direct inversion of the original operator A in (1). In constructing B we start from the additive representation of the operator A_0 is the symmetric part of the operator A.

$$A_0 = R_1 + R_2, \quad R_1 = R_2^* \tag{3}$$

Also here and below we use the skew-symmetric part of the operator A: $A_1 = \dfrac{A - A^*}{2}$.

By virtue of (3) $(Ay, y) = (A_0 y, y) = 2(R_1 y, y) = 2(R_2 y, y)$. Therefore, in (3) $R_1 > 0, R_2 > 0$. Suppose that in (2)

$$B = (D + \omega R_1)D^{-1}(D + \omega R_2), \; D = D^* > 0, \; \omega > 0, y \in H, \tag{4}$$

where D is an operator.

Because $A_0 = A_0^* > 0$, then $B = B^* > 0$. Relations (2)-(4) specify a modified alternate-triangular method (MATM) for solving the problem if operators R_1, R_2 and the methods for determining the parameters τ, ω and operator D.

The variational optimization of MATM. In the nonstationary MATM of the variational type:

$$B\frac{x^{m+1} - x^m}{\tau_{m+1}} + Ax^m = f \tag{5}$$

iterative parameters τ_{m+1} is chosen for reasons of minimization [1, 4]:

$$(\tilde{D}z^{m+1}, z^{m+1}) \to \min, \quad \tilde{D} = \tilde{D}^* > 0. \tag{6}$$

Criterion (6) together with $z^{m+1} = z^m - \tau_{m+1}B^{-1}Az^m$, gives

$$(\tilde{D}z^m - \tau_{m+1}\tilde{D}B^{-1}Az^m, z^m - \tau_{m+1}B^{-1}Az^m) \to \min \tag{7}$$

or

$$(\tilde{D}z^m, z^m) - \tau_{m+1}(\tilde{D}B^{-1}Az^m, z^m) - \tau_{m+1}(\tilde{D}z^m, B^{-1}Az^m) + \tau_{m+1}^2(\tilde{D}B^{-1}Az^m, B^{-1}Az^m) \to \min.$$

The minimum is reached when

$$\tau_{m+1} = \frac{(\tilde{D}B^{-1}Az^m, z^m) + (\tilde{D}z^m, B^{-1}Az^m)}{2(\tilde{D}B^{-1}Az^m, B^{-1}Az^m)} \tag{8}$$

Iterative methods of variational type (6), (8), for which $\tilde{D} = A^*B^{-1}A$ give:

$$\tau_{m+1} = \frac{(A^*B^{-1}AB^{-1}Az^m, z^m) + (A^*B^{-1}Az^m, B^{-1}Az^m)}{2(A^*B^{-1}AB^{-1}Az^m, B^{-1}Az^m)} =$$

$$= \frac{(AB^{-1}Az^m, B^{-1}Az^m) + (A^*B^{-1}Az^m, B^{-1}Az^m)}{2(B^{-1}AB^{-1}Az^m, AB^{-1}Az^m)} =$$

$$= \frac{(AB^{-1}Az^m, B^{-1}Az^m) + (AB^{-1}Az^m, B^{-1}Az^m)}{2(B^{-1}AB^{-1}Az^m, AB^{-1}Az^m)} = \frac{(AB^{-1}Az^m, B^{-1}Az^m)}{(B^{-1}AB^{-1}Az^m, AB^{-1}Az^m)}$$

Then the iterative parameters for MATM of minimum corrections are calculated by the formula:

$$\tau_{m+1} = \frac{\left(Aw^m, w^m\right)}{\left(B^{-1}Aw^m, Aw^m\right)}, \quad Bw^m = Ax^m - f \quad m = 0,1,\ldots. \tag{9}$$

When $A = A^* > 0$, iterative parameters τ_{m+1} can be calculated according to the method of speedy descent ($\tilde{D} = A$): $\tau_{m+1} = \dfrac{\left(w^m, r^m\right)}{\left(Aw^m, w^m\right)}, \quad m = 0,1,\ldots.$

In [10] was suggested to select a step at each iteration τ_{m+1}, uniformly distributed on the interval $\left[\tau_2, \tau_1\right]$, where τ_1 is the iterative step, calculated on the basis of the method of early descent τ_2 minimum corrections (residuals).

The adaptive optimization of MATM minimum corrections. An essential element in this approach is the additional a priori information about the initial problem (1). For MATM, this information is associated with estimates δ and Δ: $D \le \frac{1}{\delta}A_0$, $R_1D^{-1}R_2 \le \frac{\Delta}{4}A_0$.

Let us write down the MATM parameter estimates taking into account the notation introduced::

$$B = (D + \omega R_1)D^{-1}(D + \omega R_2) = (D - \omega R_1)D^{-1}(D - \omega R_2) + 2\omega A_0,$$

$$\left(By, y\right) = \left\|\left(D - \omega R_2\right)y\right\|^2_{D^{-1}} + 2\omega\left(A_0y, y\right) \ge 2\omega\left(A_0y, y\right).$$

We obtain an upper bound for the scalar product: $\left(A_0y, y\right) \le \frac{1}{2\omega}\left(By, y\right)$, using a replacement

$C_0 = B^{-1/2}A_0B^{-1/2}$, we obtain: $\gamma_2(C_0) = \dfrac{1}{2\omega}$ (10)

We estimate $\left(A_0 y, y\right)$ bottom:

$$B=(D+\omega R_1)D^{-1}(D+\omega R_2)=D+\omega A_0+\omega^2 R_1 D^{-1}R_2 \le \left(\frac{1}{\delta}+\omega+\frac{\omega^2}{4}\Delta\right)A_0,$$

$$\frac{1}{\left(\frac{1}{\delta}+\omega+\frac{\omega^2}{4}\Delta\right)}\left(By,y\right)\le\left(A_0 y,y\right) \;\Rightarrow\; \gamma_1(C_0)=\frac{\delta}{\left(1+\omega\delta+\frac{\omega^2}{4}\Delta\delta\right)}. \qquad (11)$$

The rate of convergence of the iterative methods depends on the number of conditionality of the operator C_0:

$$\nu(C_0)=\frac{\lambda_{\max}\left(C_0\right)}{\lambda_{\min}\left(C_0\right)}=\frac{\lambda_{\max}\left(B^{-1/2}A_0B^{-1/2}\right)}{\lambda_{\min}\left(B^{-1/2}A_0B^{-1/2}\right)}. \qquad (12)$$

We now note that for the MATM the estimate of the operator (10) $2\omega A_0 \le B$ is unimprovable [2]. Given this fact, we get an estimate of the number of conditionality:

$$\nu(C_0)=\frac{\lambda_{\max}\left(B^{-1/2}A_0B^{-1/2}\right)}{\lambda_{\min}\left(B^{-1/2}A_0B^{-1/2}\right)}\le\frac{1}{2\omega\lambda_{\min}\left(B^{-1/2}A_0B^{-1/2}\right)}=\max_{y\ne0}\frac{(y,y)}{2\omega\left(B^{-1/2}A_0B^{-1/2}y,y\right)}=$$

$$=\max_{y\ne0}\frac{(By,y)}{2\omega\left(A_0y,y\right)}=\max_{y\ne0}\frac{\left(\left(D+\omega A_0+\omega^2R_1D^{-1}R_2\right)y,y\right)}{2\omega\left(A_0y,y\right)}=$$

$$=\max_{y\ne0}\frac{\left(Dy,y\right)+\omega\left(A_0y,y\right)+\omega^2\left(D^{-1}R_2y,R_2y\right)}{2\omega\left(A_0y,y\right)}=$$

$$=\max_{y\ne0}\left(\frac{\left(Dy,y\right)}{2\omega\left(A_0y,y\right)}+\frac{1}{2}+\frac{\omega\left(D^{-1}R_2y,R_2y\right)}{2\left(A_0y,y\right)}\right)=\max_{y\ne0}\left(\frac{1}{2}+\frac{\sqrt{\left(Dy,y\right)\left(D^{-1}R_2y,R_2y\right)}}{2\left(A_0y,y\right)}\cdot\right.$$

$$\left.\cdot\left[\frac{1}{\omega}\sqrt{\frac{\left(Dy,y\right)}{\left(D^{-1}R_2y,R_2y\right)}}+\omega\sqrt{\frac{\left(D^{-1}R_2y,R_2y\right)}{\left(Dy,y\right)}}\right]\right).$$

Its value is minimal when: $\omega=\sqrt{\dfrac{\left(Dy,y\right)}{\left(D^{-1}R_2y,R_2y\right)}}$. (13)

Thus, we have obtained an estimate for the optimal value of the parameter ω. With allowance for (13), we find the estimate of the condition number:

$$\nu(C_0) \le \max_{y\neq 0}\left(\frac{1}{2}+\frac{\sqrt{(Dy,y)\left(D^{-1}R_2y,R_2y\right)}}{(A_0y,y)}\right) \le \frac{1}{2}\left(1+\sqrt{\frac{\Delta}{\delta}}\right)=\frac{1+\sqrt{\xi}}{2\sqrt{\xi}},\ \xi=\frac{\delta}{\Delta},$$

The vectory in (13) is defined as follows: $y = w^m$, where w^m is the correction vector. As an operator we can take the diagonal part of the matrix A. Optimal operator D is constructed from the condition for maximizing the relation $\xi = \delta/\Delta$, which determines the number of iterations. In [4] the technique of calculating the operator D for solving thePoisson equation . In [5] this result is generalized to the case of the presence of a linear source function.

The improved iterative MATM. We rewrite the system of difference equations (1) in the form with homogeneous boundary conditions, changing the right-hand side accordingly [4].

$$Ay = u,\ Ay = -\sum_{\alpha=1}^{3}\left(a_\alpha^{+1} y_{x_\alpha}\right)_{\bar{x}_\alpha} + qy,\ x\in\omega, \tag{14}$$

$$y(x) = 0,\ x\in\gamma, \tag{15}$$

$$u(x) = \varphi(x) + \varphi_1(x)/h_1^2 + \varphi_2(x)/h_2^2 + \varphi_3(x)/h_3^2,\quad x\in\omega,$$

Where $\varphi_1(x) = \begin{cases} a_1(h_1,x_2,x_3)\cdot\mu(0,x_2,x_3),\ x_1 = h_1,\\ 0,\quad 2h_1 \le x_1 \le l_1 - 2h_1,\\ a_1(l_1,x_2,x_3)\cdot\mu(l_1,x_2,x_3),\ x_1 = l_1 - h_1,\end{cases}\qquad x_2\in\omega_2,\quad x_3\in\omega_3;$

$$\varphi_2(x) = \begin{cases} a_2(x_1,h_2,x_3)\cdot\mu(x_1,0,x_3),\ x_2 = h_2,\\ 0,\quad 2h_2 \le x_2 \le l_2 - 2h_2,\\ a_2(x_1,l_2,x_2)\cdot\mu(x_1,l_2,x_3),\ x_2 = l_2 - h_2,\end{cases}\qquad x_1\in\omega_1,\quad x_3\in\omega_3;$$

$$\varphi_3(x) = \begin{cases} a_3(x_1,x_2,h_3)\cdot\mu(x_1,x_2,0),\ x_3 = h_3,\\ 0,\quad 2h_3 \le x_3 \le l_3 - 2h_3,\\ a_3(x_1,x_2,l_3)\cdot\mu(x_1,x_2,l_3),\ x_3 = l_3 - h_3,\end{cases}\qquad x_1\in\omega_1,\quad x_2\in\omega_2.$$

Source function q, a member of the equation (14) is δ-figurative character and in the overwhelming majority of grid nodes is equal to 0.

We represent the scheme of the iterative two-layer modified alternating-triangular method in the form:

$$(D+\omega R_1)D^{-1}(D+\omega R_2)\frac{y^{n+1}-y^n}{\tau_{n+1}} + Ay^n = f, \tag{16}$$

where $R_1 y = \sum_{\alpha=1}^{3}\left(\frac{a_\alpha}{h_\alpha} y_{\bar{x}_\alpha} + \frac{a_{\alpha x_\alpha}}{2h_\alpha} y + \frac{1}{6} qy\right),$ (17)

$$R_2 y = -\sum_{\alpha=1}^{3}\left(\frac{a_\alpha^{+1}}{h_\alpha} y_{x_\alpha} + \frac{a_{\alpha x_\alpha}}{2h_\alpha} y - \frac{1}{6} q y\right). \quad (18)$$

It can be seen that the operators introduced in accordance with equalities (17) and (18) are conjugate on the set of grid functions vanishing at the boundary of the grid: $R_1 = R_2^*, \quad A = R_1 + R_2$.

Estimates are obtained for the constants Δ and δ: $\delta D \le R_1 + R_2$, $R_1 D^{-1} R_2 \le \frac{\Delta}{4}(R_1 + R_2)$, (19)

and net function d(x), determines the elements of the diagonal matrix D.

$$d(x) = \sum_{\alpha=1}^{3}\left(a_\alpha^{+1} + h_\alpha^2 \chi_\alpha \left|\frac{a_{\alpha x_\alpha}}{2h_\alpha} - \frac{q}{6}\right|\right)\frac{\theta_\alpha}{h_\alpha^2}, \quad x \in \omega. \quad (20)$$

We assume that $v^\alpha(x)$ is the solution of the boundary value problem

$$\left(a_\alpha v_{\bar{x}_\alpha}^\alpha\right)_{x_\alpha} - \frac{q}{3} v^\alpha = -\left|\frac{a_{\alpha x_\alpha}}{2h_\alpha} - \frac{q}{6}\right|, \quad \alpha = 1,2,3; \quad x_\alpha \in \omega_\alpha, \quad (21)$$

$$v^\alpha = 0, \quad x_\alpha = 0, \quad x_\alpha = l_\alpha.$$

We set $c_\alpha(x_\beta, x_\delta) = \max_{x_\alpha \in \omega_\alpha} v^\alpha(x), \quad x = (x_1, x_2, x_3) \in \omega,$ (22)

where $c_\alpha(x_\beta, x_\delta) = \begin{cases} c_1(x_2, x_3), & (x_2, x_3) \in \omega_2 \times \omega_3, \quad \text{if } \alpha = 1; \\ c_2(x_1, x_3), & (x_1, x_3) \in \omega_1 \times \omega_3, \quad \text{if } \alpha = 2; \\ c_3(x_1, x_2), & (x_1, x_2) \in \omega_1 \times \omega_2, \quad \text{if } \alpha = 3. \end{cases}$

Let us now consider another set of three-point difference problems:

$$\left(a_\alpha w_{\bar{x}_\alpha}^\alpha\right)_{x_\alpha} - \frac{q}{3} w^\alpha = -\frac{a_\alpha^{+1}}{h_\alpha^2}, \quad w^\alpha = 0, x_\alpha = 0, x_\alpha = l_\alpha, \quad \alpha = 1,2,3; \quad (23)$$

$$x_\alpha \in \omega_\alpha, \quad x = (x_1, x_2, x_3) \in \omega.$$

Let $b_\alpha(x_\beta, x_\delta) = \max_{x_\alpha \in \omega_\alpha} w^\alpha(x), \quad x = (x_1, x_2, x_3) \in \omega, \quad \alpha = 1,2,3,$ (24)

where $b_\alpha(x_\beta, x_\delta) = \begin{cases} b_1(x_2, x_3), & (x_2, x_3) \in \omega_2 \times \omega_3, \quad \text{if } \alpha = 1; \\ b_2(x_1, x_3), & (x_1, x_3) \in \omega_1 \times \omega_3, \quad \text{if } \alpha = 2; \\ b_3(x_1, x_2), & (x_1, x_2) \in \omega_1 \times \omega_2, \quad \text{if } \alpha = 3; \end{cases}$ $(\beta, \delta) = \begin{cases} (2,3), \text{if} & \alpha = 1; \\ (1,3), \text{if} & \alpha = 2; \\ (1,2), \text{if} & \alpha = 3. \end{cases}$

then $\left(d \overset{o}{y}{}^2, 1\right) \le \left(A \overset{o}{y}, \overset{o}{y}\right), \quad \delta \equiv 1.$

We set $\bar{q} = \max_{(x_1, x_2, x_3) \in \omega} \{q(x_1, x_2, x_3)\},$ (25)

$$\gamma_1=\frac{4}{h_1^2}\max\left\{a_1(0,x_2,x_3),a_1(l_1,x_2,x_3),\max_{0\le x_1\le l_1-h_1}\left(\frac{a_1(x_1,x_2,x_3)+a_1(x_1+h_1,x_2,x_3)}{2}\right)\right\},$$

$$\gamma_2=\frac{4}{h_2^2}\max\left\{a_2(x_1,0,x_2),a_1(x_1,l_2,x_3),\max_{0\le x_2\le l_2-h_2}\left(\frac{a_2(x_1,x_2,x_3)+a_2(x_1,x_2+h_2,x_3)}{2}\right)\right\},$$

$$\gamma_3=\frac{4}{h_3^2}\max\left\{a_3(x_1,x_2,0),a_3(x_1,x_2,\mathrm{l}_3),\max_{0\le x_3\le l_3-h_3}\left(\frac{a_3(x_1,x_2,x_3)+a_3(x_1,x_2,x_3+h_3)}{2}\right)\right\}.$$

We introduce constant quantities (coefficients) in accordance with the equalities:

$$k_{1\alpha}=\gamma_\alpha/(\gamma_\alpha+\bar{q}/3),\quad \alpha=1,2,3. \tag{26}$$

The expression for the definition of the function d(x) is written as:

$$d(x)=\sum_{\alpha=1}^{3}\left(\frac{a_\alpha^{+1}}{h_\alpha^2}+\left(\frac{b_\alpha}{k_{1\alpha}}\right)^{\frac{1}{2}}\left|\frac{a_{\alpha x_\alpha}}{2h_\alpha}-\frac{q}{6}\right|\right)\left(b_\alpha+c_\alpha\left(\frac{b_\alpha}{k_{1\alpha}}\right)^{\frac{1}{2}}\right)^{-1}. \tag{27}$$

Estimate for the parameter Δ has the form

$$\Delta=\max_{\alpha=1,2,3}\left[\max_{x_\beta\in\omega_\beta,x_\delta\in\omega_\delta}\left(\left(b_\alpha(x_\beta,x_\delta)k_{1\alpha}(x_\beta,x_\delta)\right)^{\frac{1}{2}}+c_\alpha(x_\beta,x_\delta)\right)^2\right],\quad (\beta,\delta)=\begin{cases}(2,3),\ if & \alpha=1,\\ (1,3),\ if & \alpha=2,\\ (1,2),\ if & \alpha=3.\end{cases}$$

$\alpha=1,2,3,\quad x_\beta\in\omega_\beta,\quad x_\delta\in\omega_\delta,$

Taking into account equality $\delta=1$, we arrive at the equation defining the parameter ω_0:

$$\omega_0=2/\sqrt{\Delta}. \tag{28}$$

In the case of using the Chebyshev acceleration of the iterative process for the number of iterations $n_0(\varepsilon)$, required to achieve the specified accuracy ε the following estimate is:

$$n_0(\varepsilon)=\frac{\sqrt[4]{\Delta}\ln\left(\frac{2}{\varepsilon}\right)}{2\sqrt{2}},\ n_0(\varepsilon)=O\left(\sqrt{N_0}\ln\left(\frac{2}{\varepsilon}\right)\right),\quad N_0=\max\{N_1,N_2,N_3\}. \tag{29}$$

Convergence of MATM minimum corrections. If $x^m-x=z^m$ is the vector of error, $w^m=B^{-1}Az^m$ is the correction vector, then from equations (1), (2) we obtain:

$$B(\omega)z^{m+1}=B(\omega)z^m-\tau Az^m, \tag{30}$$

this expression can be written in the following form: $w^{m+1}=w^m-\tau B^{-1}Aw^m$,

using a replacement $v^m=B^{1/2}w^m, C=B^{-1/2}AB^{-1/2}$, we obtain: $B^{-1/2}v^{m+1}=B^{-1/2}v^m-\tau B^{-1}AB^{-1/2}v^m$ or

$$v^{m+1}=v^m-\tau Cv^m=(E-\tau C)v^m. \tag{31}$$

We estimate $\left\|v^{m+1}\right\|$:

$$\left\|v^{m+1}\right\|=\left\|(E-\tau C)v^m\right\|=\left\|\left((\theta E-\tau C_0)+\left((1-\theta)E-\tau C_1\right)\right)v^m\right\|,$$

where $C=C_0+C_1$, $C_0=C_0^*$, $C_1=-C_1^*$.

We use the triangle inequality:

$$\left\|v^{m+1}\right\|\le\theta\left\|\left(E-\frac{\tau}{\theta}C_0\right)v^m\right\|+\left\|\left((1-\theta)E-\tau C_1\right)v^m\right\|. \tag{32}$$

We estimate the first term on the right-hand side of (32) (32):

$$\left\|\left(E-\frac{\tau}{\theta}C_0\right)v^m\right\|^2=\left\|\left(E-\frac{\tau}{\theta}C_0\right)B^{1/2}w^m\right\|^2=\left(\left(B^{1/2}-\frac{\tau}{\theta}B^{-1/2}A_0\right)w^m,\left(B^{1/2}-\frac{\tau}{\theta}B^{-1/2}A_0\right)w^m\right)=$$

$$=\left(B^{1/2}w^m,B^{1/2}w^m\right)-2\frac{\tau}{\theta}\left(A_0w^m,w^m\right)+\left(\frac{\tau}{\theta}\right)^2\left(A_0w^m,B^{-1}A_0w^m\right).$$

Taking into account that the minimum ratio τ/θ is achieved with $\dfrac{\tau}{\theta}=\dfrac{\left(A_0w^m,w^m\right)}{\left(B^{-1}A_0w^m,A_0w^m\right)}$, we obtain a version of the method of minimum corrections and the estimate is written:

$$\left\|\left(E-\frac{\tau}{\theta}C_0\right)v^m\right\|^2=\left(B^{1/2}w^m,B^{1/2}w^m\right)-2\frac{\left(A_0w^m,w^m\right)}{\left(B^{-1}A_0w^m,A_0w^m\right)}\left(A_0w^m,w^m\right)+$$

$$+\left(\frac{\left(A_0w^m,w^m\right)}{\left(B^{-1}A_0w^m,A_0w^m\right)}\right)^2\left(A_0w^m,B^{-1}A_0w^m\right)=\left(B^{1/2}w^m,B^{1/2}w^m\right)-\frac{\left(A_0w^m,w^m\right)^2}{\left(B^{-1}A_0w^m,A_0w^m\right)}=$$

$$=\left(v^m,v^m\right)-\frac{\left(A_0B^{-1/2}v^m,B^{-1/2}v^m\right)^2}{\left(B^{-1}A_0B^{-1/2}v^m,A_0B^{-1/2}v^m\right)}=\left(v^m,v^m\right)-\frac{\left(B^{-1/2}A_0B^{-1/2}v^m,v^m\right)^2}{\left(B^{-1/2}A_0B^{-1/2}v^m,B^{-1/2}A_0B^{-1/2}v^m\right)}=$$

$$=\left(v^m,v^m\right)-\frac{\left(C_0v^m,v^m\right)^2}{\left(C_0v^m,C_0v^m\right)}=\left(1-\frac{\left(C_0v^m,v^m\right)^2}{\left(C_0v^m,C_0v^m\right)\left(v^m,v^m\right)}\right)\left\|v^m\right\|^2.$$

As a result we get: $$\left\|\left(E-\frac{\tau}{\theta}C_0\right)v^m\right\|=\sqrt{1-\frac{\left(C_0v^m,v^m\right)^2}{\left(C_0v^m,C_0v^m\right)\left(v^m,v^m\right)}}\left\|v^m\right\|. \tag{33}$$

For the second term in (32), taking $\left(C_1v^m,v^m\right)=0$ we have:

$$\left\|\left((1-\theta)E-\tau C_1\right)v^m\right\|^2=\left((1-\theta)v^m,(1-\theta)v^m\right)-\left(\tau C_1v^m,(1-\theta)v^m\right)+$$

$$+\left(\tau C_1v^m,\tau C_1v^m\right)=(1-\theta)^2\left(v^m,v^m\right)+\theta^2\left(\frac{\tau}{\theta}C_1v^m,\frac{\tau}{\theta}C_1v^m\right). \tag{34}$$

The substitution of (33), (34) into (32) gives:

$$\left\|v^{m+1}\right\|\le\theta\left\|\left(E-\frac{\tau}{\theta}C_0\right)v^m\right\|+\left\|\left((1-\theta)E-\tau C_1\right)v^m\right\|=$$

$$=\theta\sqrt{\left(1-\frac{\left(C_0v^m,v^m\right)^2}{\left(C_0v^m,C_0v^m\right)\left(v^m,v^m\right)}\right)}\left\|v^m\right\|+\sqrt{(1-\theta)^2\left(v^m,v^m\right)+\theta^2\left(\frac{\tau}{\theta}C_1v^m,\frac{\tau}{\theta}C_1v^m\right)}=$$

$$=\theta\sqrt{\left(1-\frac{\left(C_0v^m,v^m\right)^2}{\left(C_0v^m,C_0v^m\right)\left(v^m,v^m\right)}\right)}\left\|v^m\right\|+\sqrt{(1-\theta)^2+\theta^2\frac{\left(\frac{\tau}{\theta}C_1v^m,\frac{\tau}{\theta}C_1v^m\right)}{\left(v^m,v^m\right)}}\left\|v^m\right\|.$$

For convenience, we introduce the notation:

$$s=\sqrt{\left(1-\frac{\left(C_0v^m,v^m\right)^2}{\left(C_0v^m,C_0v^m\right)\left(v^m,v^m\right)}\right)},\quad \gamma=\frac{\left(\frac{\tau}{\theta}C_1v^m,\frac{\tau}{\theta}C_1v^m\right)}{\left(v^m,v^m\right)},$$

$$\tau=\frac{\left(A_0\omega^m,\omega^m\right)\theta}{\left(B^{-1}A_0\omega^m,A_0\omega^m\right)}. \tag{35}$$

Evaluation $\left\|v^{m+1}\right\|$ with allowance for (35) takes the form:

$$\left\|v^{m+1}\right\|\le\left(\theta s+\sqrt{(1-\theta)^2+\theta^2\gamma}\right)\left\|v^m\right\|=\left(\theta s+\sqrt{1-2\theta+\theta^2(1+\gamma)}\right)\left\|v^m\right\|.$$

We introduce the change of variables $\theta=\dfrac{1-\eta}{1+\gamma}$:

$$\left\|v^{m+1}\right\|\le\left(\frac{1-\eta}{1+\gamma}s+\sqrt{\frac{1+\gamma-2+2\eta+(1-\eta)^2}{1+\gamma}}\right)\left\|v^m\right\|=$$

$$=\left(\frac{1-\eta}{1+\gamma}s+\sqrt{\frac{1+\gamma-2+2\eta+1-2\eta+\eta^2}{1+\gamma}}\right)\left\|v^m\right\|,$$

$$\left\|v^{m+1}\right\|\le\left(\frac{1-\eta}{1+\gamma}s+\sqrt{\frac{\gamma+\eta^2}{1+\gamma}}\right)\left\|v^m\right\|. \tag{36}$$

Let us find the optimal parameter η for this we take the derivative of the right-hand side of (36):

$$\left(\frac{1-\eta}{1+\gamma}s+\sqrt{\frac{\gamma+\eta^2}{1+\gamma}}\right)'_\eta=-\frac{s}{1+\gamma}+\frac{\eta}{\sqrt{(1+\gamma)(\gamma+\eta^2)}}=0. \qquad (37)$$

Optimal η is equal to: $\eta=\sqrt{\dfrac{s^2\gamma}{(1+\gamma-s^2)}}$ (38)

in view of (37) and $\left(\dfrac{1-\gamma}{1+k}s+\sqrt{\dfrac{\gamma+\eta^2}{1+\gamma}}\right)''_{\eta\eta}>0$.

We substitute (38) into (36):

$$\left\|v^{m+1}\right\|\le\left(\frac{1-\sqrt{\dfrac{s^2\gamma}{(1+\gamma-s^2)}}}{1+\gamma}s+\sqrt{\frac{\gamma+\dfrac{s^2\gamma}{(1+\gamma-s^2)}}{1+\gamma}}\right)\left\|v^m\right\|=$$

$$=\left(\frac{1-\sqrt{\dfrac{s^2\gamma}{(1+\gamma-s^2)}}}{1+\gamma}s+\sqrt{\frac{\gamma(1+\gamma-s^2)+s^2\gamma}{(1+\gamma)(1+\gamma-s^2)}}\right)\left\|v^m\right\|=$$

$$=\left(\frac{s-s^2\sqrt{\dfrac{\gamma}{(1+\gamma-s^2)}}}{1+\gamma}+\sqrt{\frac{\gamma}{(1+\gamma-s^2)}}\right)\left\|v^m\right\|=$$

$$=\left(\frac{s+(1+\gamma-s^2)\sqrt{\dfrac{\gamma}{(1+\gamma-s^2)}}}{1+\gamma}\right)\left\|v^m\right\|=\left(\frac{s+\sqrt{\gamma(1+\gamma-s^2)}}{1+\gamma}\right)\left\|v^m\right\|.$$

Thus, we obtain the estimate of convergence:

$$\rho\le\left(\frac{s+\sqrt{\gamma(1+\gamma-s^2)}}{1+\gamma}\right). \qquad (39)$$

Rate of convergence MATM ($v^m = B^{1/2}w^m, C_0 = B^{-1/2}A_0B^{-1/2}$) depends on:

$$s=\sqrt{\left(1-\frac{\left(C_0v^m,v^m\right)^2}{\left(C_0v^m,C_0v^m\right)\left(v^m,v^m\right)}\right)}=\sqrt{\left(1-\frac{\left(A_0w^m,w^m\right)^2}{\left(A_0B^{-1}A_0w^m,w^m\right)\left(Bw^m,w^m\right)}\right)}.$$

We use the inequality $xy\le\left(ax+y/a\right)^2/4$. This inequality holds for all a.

$$s^2\le1-\frac{4\left(A_0w^m,w^m\right)^2}{\left(a\left(A_0B^{-1}A_0w^m,w^m\right)+\left(Bw^m,w^m\right)/a\right)^2}=$$

$$=1-\frac{4}{\left(a\left(C_0v^m,C_0v^m\right)/\left(C_0v^m,v^m\right)+\left(Bw^m,w^m\right)/a\left(A_0w^m,w^m\right)\right)^2}.$$

We now note that for the MATM the estimate of the operator (10) $2\omega A_0\le B$ is unimprovable [2].

$$s^2\le1-\frac{4}{\left(a/2\omega+\left(Bw^m,w^m\right)/a\left(A_0w^m,w^m\right)\right)^2}.$$

As a parameter a take $2\omega b$ $s^2\le1-\dfrac{4}{\left(b+\dfrac{\left(Bw^m,w^m\right)}{2\omega b\left(A_0w^m,w^m\right)}\right)^2}$.

The rate of convergence of the method depends on $\left(Bw^m,w^m\right)/\left(2\omega A_0w^m,w^m\right)$. Let us estimate this expression

$$\frac{\left(Bw^m,w^m\right)}{2\omega\left(A_0w^m,w^m\right)}=\frac{\left(\left(D+\omega A_0+\omega^2R_1D^{-1}R_2\right)w^m,w^m\right)}{2\omega\left(A_0w^m,w^m\right)}=$$

$$=\left(\frac{\left(Dw^m,w^m\right)}{2\omega\left(A_0w^m,w^m\right)}+\frac{1}{2}+\frac{\omega\left(D^{-1}R_2w^m,R_2w^m\right)}{2\left(A_0w^m,w^m\right)}\right)=\left(\frac{1}{2}+\frac{\sqrt{\left(Dw^m,w^m\right)\left(D^{-1}R_2w^m,R_2w^m\right)}}{2\left(A_0w^m,w^m\right)}\times\right.$$

$$\left.\times\left[\frac{1}{\omega}\sqrt{\frac{\left(Dw^m,w^m\right)}{\left(D^{-1}R_2w^m,R_2w^m\right)}}+\omega\sqrt{\frac{\left(D^{-1}R_2w^m,R_2w^m\right)}{\left(Dw^m,w^m\right)}}\right]\right).$$

Value ω is minimal in case $\omega=\sqrt{\dfrac{\left(Dw^m,w^m\right)}{\left(D^{-1}R_2w^m,R_2w^m\right)}}$.

Thus, the optimal y in the formula (39) is given by $y = w^m$, wherer w^m is the correction vector. We obtained an estimate for the optimal value of the parameter ω.

$$\nu = \max_{y\neq 0}\left(\frac{1}{2}+\frac{\sqrt{(Dy,y)(D^{-1}R_2y,R_2y)}}{(A_0y,y)}\right)\le \frac{1}{2}\left(1+\sqrt{\frac{\Delta}{\delta}}\right)=\frac{1+\sqrt{\xi}}{2\sqrt{\xi}},\ \xi=\frac{\delta}{\Delta}.$$

Take $b=\nu^{1/2}$, it is easy to verify that the expression on the right-hand side of (39) is less than unity for $s<1$. For s and γ the following estimates hold:

$$s\le\sqrt{1-\frac{4}{\left(\nu^{1/2}+\nu^{-1/2}\right)^2}}=\frac{\nu-1}{\nu+1}=s_{\max}; \tag{40}$$

$$\gamma=\frac{\left(B^{-1}A_1w^m,A_1w^m\right)}{\left(B^{-1}A_0w^m,A_0w^m\right)}\left(1-s^2\right)\le\frac{\left(B^{-1}A_1w^m,A_1w^m\right)}{\left(B^{-1}A_0w^m,A_0w^m\right)},$$

where ν is the number of matrix conditioning C_0.

We introduce the notation $k=\dfrac{\left(B^{-1}A_1w^m,A_1w^m\right)}{\left(B^{-1}A_0w^m,A_0w^m\right)},\ \gamma=k\left(1-s^2\right).$ (41)

We take the derivative of s from the right side of (39) with allowance for (41):

$$\left(\frac{s+\left(1-s^2\right)\sqrt{k\left(k+1\right)}}{1+k\left(1-s^2\right)}\right)'_s=\frac{(1+k)-2s\sqrt{k(k+1)}+ks^2}{\left(1+k\left(1-s^2\right)\right)^2}\ge 0. \tag{42}$$

The estimate (39), taking (42) into account, is written as:

$$\rho\le\left(\frac{s_{\max}+\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}}{1+\gamma}\right). \tag{43}$$

We multiply the numerator and denominator of expression (43) by $\left(s_{\max}\gamma+\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}\right)\Big/(1+\gamma)$, as a result we get: $\rho\le s_{\max}\dfrac{\gamma/s_{\max}+\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}}{s_{\max}\gamma+\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}}$.

In view of (40), we obtain: $\rho\le\dfrac{\nu\dfrac{\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}+\gamma}{\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}-\gamma}-1}{\nu\dfrac{\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}+\gamma}{\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}-\gamma}+1}$ or $\rho\le\dfrac{\nu^*-1}{\nu^*+1}$, (44) where $\nu^* = \nu \dfrac{\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}+\gamma}{\sqrt{\gamma\left(1+\gamma-s_{\max}{}^2\right)}-\gamma}$. (45)

Expression (45) taking into account (41), (42) takes the form: $\nu^* = \nu\left(\sqrt{1+k}+\sqrt{k}\right)^2$, (46)

where ν is the number of matrix conditioning C_0.

For monotone circuits [3]: $Pe < 2$ (Pe is the grid Peclet number) there is a restriction $k < 1$. We obtain the estimate of the parameter ν^*: $\nu^* < \nu\left(3+2\sqrt{2}\right)$.

Thus, the algorithm for calculating the grid equations can be written as:

$$r^m = Ax^m - f, \tag{47}$$

$$B(\omega_m)w^m = r^m, \tag{48}$$

$$\tilde{\omega}_m = \sqrt{\frac{\left(Dw^m, w^m\right)}{\left(D^{-1}R_2 w^m, R_2 w^m\right)}}, \tag{49}$$

$$s_m{}^2 = 1 - \frac{\left(A_0 w^m, w^m\right)^2}{\left(B^{-1}A_0 w^m, A_0 w^m\right)\left(Bw^m, w^m\right)}, \tag{50}$$

$$k_m = \frac{\left(B^{-1}A_1 w^m, A_1 w^m\right)}{\left(B^{-1}A_0 w^m, A_0 w^m\right)}, \tag{51}$$

$$\theta_m = \frac{1-\sqrt{\dfrac{s_m{}^2 k_m}{(1+k_m)}}}{1+k_m\left(1-s_m{}^2\right)}, \tag{52}$$

$$\tau_{m+1} = \theta_m \frac{\left(A_0 w^m, w^m\right)}{\left(B^{-1}A_0 w^m, A_0 w^m\right)}, \tag{53}$$

$$x^{m+1} = x^m - \tau_{m+1} w^m, \tag{54}$$

$$\omega_{m+1} = \tilde{\omega}_m. \tag{55}$$

Conclusion. In this paper, a variant of the modified alternating-triangular iterative method of minimal corrections for solving grid equations with a nonselfadjoint operator is constructed, and estimates of the rate of convergence are performed. It follows from the obtained estimate that the rate of convergence of the iterative methods depends on the number of conditionality of the symmetric part of the operator A. This fact served as the basis for constructing an alternate triangular method that showed its effectiveness for moderate Peclet numbers. We note that the developed version of the method of minimum corrections (53) coincides with the classical version of this method (9) in the self-adjoint case.

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Адаптивный модифицированный попеременно-треугольный метод *

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Работа посвящена разработке адаптивных методов для решения задач конвекции-диффузии. Для данного класса задач построен вариант адаптивного модифицированного попеременно–треугольного метода минимальных поправок и аналитически получены оценки скорости сходимости для данного метода.

Ключевые слова: сеточные уравнения с несамосопряженным оператором; адаптивный попеременно-треугольный метод; параллельные алгоритмы решения сеточных уравнений

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